GRUSHKO-LIKE THEOREMS FOR A CLASS
OF FINITELY PRESENTED SEMIGROUPS

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ABSTRACT. An analogue of Grushko's theorem and the unsolvability of the isomorphism problem hold for semigroups in the class of all finitely presented semigroups that have a presentation in which there is no defining relation of the form $1 \Leftrightarrow A$; this class is not recursive but is recursively enumerable.

We will define a class of finitely presented semigroups in terms of combinatorial properties of their presentations. For that class we prove (Theorem 1) theorems similar to Grushko's theorem for finitely presented groups (see, for example, [**K**, p. 58]), and (Theorem 2) a theorem similar to a theorem of Rabin (see [**R**, Theorem 3.1]), namely, for any given semigroup in that class there is no algorithm to decide of each finitely presented semigroup whether or not it is isomorphic to the given semigroup. We also show (Theorem 3) that the class is not recursive.

We assume that the reader is familiar with the standard notions and terminology (as in [**D**] or [**Y1**]) of Thue systems, finitely presented semigroups, free products of semigroups, solvable and unsolvable problems and partial recursive functions. We will be thinking of a Thue system, denoted by T, as a finite presentation of a semigroup and denote the semigroup presented by T by S_T. In general, let $T = \langle a_1, \cdots, a_m; A_i \Leftrightarrow A'_i, 1 \leq i \leq n \rangle$ where $\{a_1, \cdots, a_m\}$ is the alphabet of T and the $A_i \Leftrightarrow A'_i$ are the defining relations. We let 1 denote the empty word. For a word A on the alphabet of T, let $[A]_T$ denote the equivalence class of all words equivalent to A by means of the defining relations of T. Thus, $[A]_T$ is an element of S_T, and in particular $\{[a_i] | 1 \leq i \leq m\}$ generates S_T and $[1]_T$ is the identity element of S_T. Rather than distinguish between a_i as a letter of the alphabet of T and $[a_i]$ as a generator of S_T, we will use the word *generator* to cover both cases. When $[A]_T = [A']_T$ we may also write $A \sim A'$ in T to emphasize the fact that there must be a derivation of A' from A (A from A') in T. Also we write $[A] = [A']$ in T to avoid the subscript.

Let card$[A]_T$ denote the number of distinct words equivalent to A in T. We assume henceforth that no Thue system contains defining relations of the form $a_i \Leftrightarrow a_j$, $a_i \Leftrightarrow 1$, or $1 \Leftrightarrow 1$. We also assume that every semigroup mentioned in this paper is finitely presented. We write $S \simeq S'$ for S isomorphic to S'. If T_1 and T_2 are Thue systems with disjoint sets of generators, then we write $T_1 * T_2$ for the Thue system which has exactly the union of the two sets of generators as its generators and the union of the two sets of defining relations as its defining relations. Then $S_{T_1 * T_2}$ is the free product of S_{T_1} and S_{T_2} which is usually written $S_{T_1} * S_{T_2}$. (For more details see [C&P, p. 140].)

Let $\mathscr{E}$ be the set of all Thue systems in which there are no defining relations of the form $1 \Leftrightarrow A$ where A is a nonempty word on the generators of the Thue system. Let

$$\hat{\mathscr{E}} = \{S \mid S \text{ is a semigroup for which there exists a } T \in \mathscr{E},$$

$$\text{such that } S \simeq S_T\}.$$

(One may also think of $\hat{\mathscr{E}}$ as $\{T \mid \text{there exists a } T' \in \mathscr{E} \text{ such that } S_T \simeq S_{T'}\}$.)

LEMMA 1. *Let* $T = \langle a_1, \cdots, a_m; A_i \Leftrightarrow A_i', 1 \leq i \leq n \rangle$ *be a Thue system. If* $T \in \mathscr{E}$ *then* card$[1]_T = 1$.

PROOF. If there were a nonempty word A' such that $[1] = [A']$ in T, then there would be a derivation of $1 \sim A'$ in T and that would require a defining relation of the form $1 \Leftrightarrow A$ for some nonempty word A.

LEMMA 2. *Let* $T_1 = \langle b_1, \cdots, b_p; B_i \Leftrightarrow B_i', 1 \leq i \leq q \rangle$ *and* $T_2 = \langle c_1, \cdots, c_r; C_i \Leftrightarrow C_i', 1 \leq i \leq s \rangle$ *be Thue systems in* $\mathscr{E}$. *Suppose* U *and* V *are words on the b's and c's such that* $[U] = [V]$ *in* $S_{T_1} * S_{T_2}$. *Then, if* U *is* $B_1 C_1 \cdots B_k C_k$ *then* V *must be* $B_1' C_1' \cdots B_k' C_k'$ *where, for* $1 \leq j \leq k$, (i) B_j, B_j' *are words on the b's and* C_j, C_j' *are words on the c's, and* (ii) $[B_j] = [B_j']$ *in* T_1 *and* B_j *is the empty word if and only if* B_j' *is, and,* $[C_j] = [C_j']$ *in* T_2 *and* C_j *is the empty word if and only if* C_j' *is.*

PROOF. Consider a derivation of $U \sim V$ in $T_1 * T_2$. Since all defining relations are either from T_1 or from T_2 there is no defining relation in which both b's and c's occur on at least one side. Further, since $T_1, T_2 \in \mathscr{E}$ there are no defining relations $1 \Leftrightarrow B$ or $1 \Leftrightarrow C$. So each step of a derivation takes place within a factor B_i or a factor C_i and cannot collapse such a factor to 1 nor insert a new factor.

LEMMA 3. *Let* $S \simeq S_1 * S_2$. *Then,* $S \in \hat{\mathscr{E}}$ *if and only if* S_1 *and* $S_2 \in \hat{\mathscr{E}}$.

PROOF. If $S_1, S_2 \in \hat{\mathscr{E}}$ then certainly $S \in \hat{\mathscr{E}}$. So suppose $S \in \hat{\mathscr{E}}$ and let $T \in \mathscr{E}$ present S. If $S_1 \notin \hat{\mathscr{E}}$, then for every Thue system which presents S_1

there must be some defining relation of the form $1 \Leftrightarrow B$ where B is a nonempty word on the generators of that Thue system. For at least one such Thue system, say T_1, and one such relation, say B is $b_{i_1} \cdots b_{i_t}$, it must be that not all b_{i_j} are equivalent to 1 in T_1. For otherwise, we could obtain, by means of a Tietze transformation (see [**MK&S**, p. 50]), a Thue system T_1' which presents S_1 in which none of the generators $b_{i_1}, \cdots, b_{i_t}$ occur nor does the defining relation $1 \Leftrightarrow B$; thus by repeating this process we could obtain a Thue system in $\mathscr{E}$ that presents S_1. So consider T_1 as specified. Let θ be the given isomorphism; we can consider $\theta^{-1}[1]_{T_1} = \theta^{-1}[B] = \theta^{-1}[b_{i_1}] \cdots \theta^{-1}[b_{i_t}]$ in S. For each b_{i_j}, $\theta^{-1}[b_{i_j}] = [A_{i_j}]$ for some word A_{i_j} on the generators of T. Since there is at least one b_{i_j} such that $[b_{i_j}] \neq [1]$ in T_1, there must be a nonempty A_{i_j} such that $\theta^{-1}[b_{i_j}] = [A_{i_j}]$ in S. But then $[1]_T = \theta^{-1}[1]_{T_1} = [A_{i_1} \cdots A_{i_j} \cdots A_{i_t}]$ which contradicts the fact, from Lemma 1, that $\text{card}[1]_T = 1$. Therefore, if $S \in \mathscr{E}$ and $S \simeq S_1 * S_2$, then $S_1, S_2 \in \hat{\mathscr{E}}$.

For a Thue system T let $|T|$ denote the number of generators of T. If S is a semigroup let rank S denote the minimal number of generators of S, i.e. there is a T such that $S_T \simeq S$ and for all T' that present S, $|T| \leq |T'|$; rank $S = |T|$.

For the semigroups in $\hat{\mathscr{E}}$ we have the following kind of analogue of Grushko's Theorem.

THEOREM 1. *Suppose that* $\theta : S \to S_1 * S_2$ *is an onto isomorphism and that* S *or,* S_1 *and* S_2, *are in* $\hat{\mathscr{E}}$. (a) *Let* $T = \langle a_1, \cdots, a_m; A_i \Leftrightarrow A_i', 1 \leq i \leq n \rangle$ *and suppose* $T \in \mathscr{E}$ *and presents* S. *If* $|T| = \text{rank } S$, *then for each* i, $1 \leq i \leq m$, $\theta[a_i]$ *is an element of* S_1 *or an element of* S_2. *Also,* (b) rank $S = $ rank $S_1 +$ rank S_2.

PROOF. (a) Let $T_1 = \langle b_1, \cdots, b_p; B_i \Leftrightarrow B_i', 1 \leq i \leq q \rangle$ and $T_2 = \langle c_1, \cdots, c_r; C_i \Leftrightarrow C_i', 1 \leq i \leq s \rangle$ present S_1 and S_2, respectively. By Lemma 3 we may assume that $T_1, T_2 \in \mathscr{E}$. The proof is by contradiction. Suppose there is an a_i such that $\theta[a_i] \in S_1 * S_2 - (S_1 \cup S_2)$. Designate those a_i for which $\theta[a_i] \in S_1$ by a_i', those for which $\theta[a_i] \in S_2$ by a_i'', and those for which $\theta[a_i] \in S_1 * S_2 - (S_1 \cup S_2)$ by $\bar{a}_i$. We claim first that for each b_i, if A_i is a word on the generators of T such that $\theta^{-1}[b_i] = [A_i]$, then in fact A_i is a word on the a_i'''s. (A similar claim is made for each c_i.) For, suppose there is a word A of the form $A_1 a_j'' A_2$ or $A_1 \bar{a}_j A_2$ such that $\theta^{-1}[b_i] = [A]$. Since $\theta[a_j''] = [C_j]$ and $\theta[\bar{a}_j] = [B_1 C_1 \cdots B_t C_t]$ where at least one B_i and one C_j are not the empty word, then there is a word W such that $\theta\theta^{-1}[b_i] = [b_i] = [W]$ and W contains both nonempty subwords on b_i's and nonempty subwords on c_i's. But, by Lemma 2 this is impossible— any W such that $[b_i] = [W]$ must be on b_i's only. (A similar argument holds for the c_i.) Now consider any $\bar{a}_i$. $\theta[\bar{a}] = [B_1 C_1 \cdots B_t C_t]$ where at

least one B_i and one C_j are nonempty words on b_i's and c_i's respectively. Then, by the claim just established there are words A_j' on the a_i''s and A_j'' on the a_i'''s such that $\theta^{-1}[B_j]=[A_j']$ and $\theta^{-1}[C_j]=[A_j'']$ for $1 \leqq i \leqq t$. Therefore

$$[\bar{a}] = \theta^{-1}\theta[\bar{a}] = \theta^{-1}[B_1]\theta^{-1}[C_1] \cdots \theta^{-1}[C_t] = [A_1'A_1'' \cdots A_t''].$$

That is, $\bar{a}$ is equivalent to a word in which $\bar{a}$ does not occur. Thus, by a Tietze transformation, there is a presentation of S with one less generator, so $|T| > \text{rank } S$.

For the proof of (b) suppose T presents S and $|T| = \text{rank } S$, and T_i presents S_i for $i = 1, 2$. Suppose T, T_1 and T_2 as given by the theorem and the proof of part (a). Since $T_1 * T_2$ presents S and $|T_1 * T_2| = |T_1| + |T_2|$, certainly rank $S \leqq |T_1| + |T_2|$. As in part (a) use a_i' and a_i'' to indicate which generators are sent by θ to S_1 and which to S_2. (We now know there are no $\bar{a}_i$.) We claim that $\{\theta[a_i'] | \text{all } a_i'\}$ generates S_1; a similar claim is made for S_2. Let B be any word on the b_i's. Then $\theta^{-1}[B] = [A]$ for some word A of the form $a_{i_1}'a_{i_2}' \cdots a_{i_n}'$. So $[B] = \theta[a_{i_1}'] \cdots \theta[a_{i_n}']$. So there are Thue systems T_1' and T_2', which present S_1 and S_2, respectively, such that $|T_1'|$ equals the number of a_i''s and $|T_2'|$ equals the number of a_i'''s. So rank $S = \text{rank } S_1 + \text{rank } S_2$. (Remark: it is easy to show that $T_1', T_2' \in \mathscr{E}$.)

COROLLARY. *If $S \in \hat{\mathscr{E}}$ and $S \simeq S * S'$ for some semigroup S', then S' is the trivial semigroup.*

PROOF. By Lemma 3, $S \in \hat{\mathscr{E}}$ implies that $S' \in \hat{\mathscr{E}}$, and by Theorem 1, rank $S = \text{rank } S + \text{rank } S'$, so rank S' must be 0.

THEOREM 2. *For any given T such that $S_T \in \hat{\mathscr{E}}$ there is no alogrithm to decide of an arbitrary Thue system T' whether or not $S_T \simeq S_{T'}$.*

PROOF. Let $T_0 \in \mathscr{E}$ such that $S_{T_0} \simeq S_T$. Consider

$\{S_{T_0} * S_{T'} \mid T'$ varies over all Thue systems with generators

disjoint from those of $T_0\}$.

Then, by the corollary, $S_{T_0} \simeq S_{T_0} * S_{T'}$ if and only if $S_{T'}$ is trivial. But it is known (see [M] or [Mo]) that there is no algorithm to decide of an arbitrary Thue system whether or not it presents the trivial semigroup.

These two theorems have a certain will-o'-the-wisp quality since, as proved below, $\hat{\mathscr{E}}$ is not a recursive set. A *Markov property* $\hat{\mathscr{P}}$ of finitely presented semigroups is a property (i.e. a class of semigroups) which is closed under isomorphism and for which there are Thue systems T_1 and T_2 such that (i) $S_{T_1} \in \hat{\mathscr{P}}$ and (ii) for any semigroup S if S_{T_2} can be embedded in S, then $S \notin \hat{\mathscr{P}}$.

LEMMA 4. $\hat{\mathscr{E}}$ is a Markov property.

PROOF. $\hat{\mathscr{E}}$ obviously satisfies (i), For (ii), consider $T=\langle a, b; ab\Leftrightarrow 1\rangle$; it does not belong to $\mathscr{E}$. We will see that S_T cannot be embedded in any S that belongs to $\hat{\mathscr{E}}$. For any S and θ such that θ embeds S_T into S we have $\theta[a]\cdot\theta[b]=[1]$ in S. Since $[a]$, $[b]$ and $[1]$ are distinct elements of S_T, for *any* presentation T' of S there must be nonempty words A and B on the generators of T' such that $\theta[a]=[A]$ and $\theta[b]=[B]$. Thus $AB\sim 1$ in T' so there must be some defining relation of the form $A_i\Leftrightarrow 1$ in T'. Therefore $T'\notin\mathscr{E}$ and so $S\notin\hat{\mathscr{E}}$.

THEOREM 3. $\hat{\mathscr{E}}$ is not recursive but is recursively enumerable.

PROOF. In [M] (or see [Mo]), Markov shows that for any Markov property $\hat{\mathscr{P}}$ there can be no algorithm to decide of an arbitrary Thue system T whether or not $S_T\in\hat{\mathscr{P}}$. On the other hand, $\mathscr{E}$ is recursive, so by recursively enumerating all Thue systems presenting semigroups isomorphic to semigroups presented by Thue systems in $\mathscr{E}$ one obtains a recursive enumeration of $\hat{\mathscr{E}}$. That one can recursively enumerate the Thue systems presenting semigroups isomorphic to a given one is generally believed; a proof is offered in [Y2].

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ADDED IN PROOF. Theorem 1 does not hold for semigroups in general; counterexample: $\langle a, b; ab=1\rangle * \langle c; c^2=1\rangle$. Also, let S_1, S_2, S be semigroups with $S_1\in\hat{\mathscr{E}}$. If X is a set of generators of minimal length w.r.t. S_1 and S_2, then for each $x\in X$ either $x\in S_1$ or $x\in S_2$.

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