A NOTE ON THE HOMEOMORPHISM GROUP OF THE RATIONAL NUMBERS

WAYNE R. PARK

Abstract. Let Q be the rational numbers with the usual topology, \( H(Q) \) the group of homeomorphisms of \( Q \), \( \gamma_c \) the convergence structure of continuous convergence, and \( \sigma \) the coarsest admissible convergence structure which makes \( H(Q) \) a convergence group. A counterexample is constructed to show that if \( \kappa \) is a convergence structure on \( H(Q) \) such that \( \gamma_c \leq \kappa \leq \sigma \), then \( \kappa \) is never principal, hence never topological.

In a previous work by the same author [4], a convergence structure \( \sigma \) on \( H(X) \), the group of homeomorphisms of the convergence space \( X \), was developed which is the coarsest of the admissible convergence group structures on \( H(X) \). When \( X \) is a locally compact topological space, the \( \sigma \) convergence structure becomes a topology, specifically the \( g \)-topology [1]. What happens to \( \sigma \) when \( X \) is no longer locally compact was left as an open question. Here a simple counterexample describes how badly non-topological the situation is when \( X = Q \), the rational numbers with the usual topology. We use the notation as given in [3] and [4].

Let \( H(X) \) represent the group of homeomorphisms of the convergence space \((X, \tau)\).

Definition. For each \( f \in H(X) \) let \( \gamma_c f \) consist of all filters \( \mathcal{F} \) on \( H(X) \) such that

1. For all \( x \in X \) and for all \( \Phi \in \tau_x \), \( \mathcal{F}(\Phi) \in \tau f(x) \).

Here \( \mathcal{F}(\Phi) = \omega(\mathcal{F} \times \Phi) \) where \( \omega : H(X) \times X \rightarrow X \) is the evaluation mapping. The convergence structure \( \gamma_c \) is called the convergence structure of continuous convergence [2].

Definition. For each \( f \in H(X) \) let \( \sigma f \) consist of all filters \( \mathcal{F} \) on \( H(X) \) such that \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) simultaneously belong to \( \gamma_c f \).

Here \( \mathcal{F}^{-1} \) is the filter with filter base \( \{F^{-1} | F \in \mathcal{F} \} \) where \( F^{-1} = \{f^{-1} \in H(X) | f \in F \} \).

It is clear that \( \sigma \) is a finer convergence structure than \( \gamma_c \) \((\gamma_c \leq \sigma)\). Moreover for \( X = Q \), the rational numbers with the usual topology, \( \sigma \) is strictly finer than \( \gamma_c \) [4]. It is known that \( \gamma_c \) on \( H(Q) \) is not a topology [1], and

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moreover is not even a principal convergence structure [2]. Here we show
the same is true of \( \sigma \) on \( H(Q) \), and in fact, true for any convergence
structure between \( \gamma \) and \( \sigma \).

**Theorem.** If \( \kappa \) is a convergence structure on \( H(Q) \) such that \( \gamma \leq \kappa \leq \sigma \),
then \( \kappa \) is not a principal convergence structure.

**Proof.** Let \( I \) be the set of irrational numbers. For each \( \alpha \in I \), define a
sequence of homeomorphisms \( f_{n,\alpha} \) in \( H(Q) \) by

\[
f_{n,\alpha}(x) = \begin{cases} -x & \text{if } |\alpha| - 1/n < |x| < |\alpha| + 1/n, \\ x & \text{otherwise,} \end{cases}
\]

where \( x \in Q \), and the integer \( n \) is larger than \( 1/|\alpha| \).

For each \( \alpha \in I \), let \( F_{N,\alpha} = \{ f_{n,\alpha} \in H(Q) | n \geq N \} \) and let \( \mathcal{F}_\alpha \) be the filter
generated by the filter base \( \{ F_{N,\alpha} \} \) where \( N \) runs through the integers
greater than \( 1/|\alpha| \). As \( f_{-n,\alpha}^{-1} = f_{n,\alpha} \), it follows that \( \mathcal{F}_\alpha = \mathcal{F}_\alpha^{-1} \) for each \( \alpha \in I \).
Moreover, \( \mathcal{F}_\alpha \in \gamma_\alpha(id) \) where \( id \) denotes the identity homeomorphism.
Namely, let \( q \in Q \) and let \( \Phi \) denote the neighborhood filter of \( q \) with the
usual topology (convergence structure) \( \tau \). To satisfy condition (1) above,
let \( U \) be arbitrary in \( \Phi \). One can find \( N \) sufficiently large and a \( V \) in \( \Phi \)
such that \( \omega(F_{N,\alpha} \times V) \subseteq U \). Hence \( \mathcal{F}_\alpha(\Phi) \subseteq \Phi \), so \( \mathcal{F}_\alpha(\Phi) \) converges
topologically to \( id(q) = q \), that is, \( \mathcal{F}_\alpha(\Phi) \in \tau id(q) \).

Now as \( \mathcal{F}_\alpha = \mathcal{F}_\alpha^{-1} \) and as \( \mathcal{F}_\alpha \in \gamma_\alpha(id) \), it follows that \( \mathcal{F}_\alpha \in \sigma(id) \) for
each \( \alpha \in I \).

But now, if \( q \) is any nonzero rational number and \( \Phi \) its neighborhood
filter, \( \omega(\bigwedge_\alpha \mathcal{F}_\alpha \times \Phi) = (\bigwedge_\alpha \mathcal{F}_\alpha)(\Phi) \) does not converge to \( id(q) = q \). If it did,
for any \( U \) in \( \Phi \) we would need to find a \( V \) in \( \Phi \) and a filter base element
\( \bigcup_\alpha F_{N,\alpha} \) in \( \bigwedge_\alpha \mathcal{F}_\alpha \) such that \( \omega(\bigcup_\alpha F_{N,\alpha} \times V) \subseteq U \). As irrationals lie
arbitrarily close to \( q \), this step is impossible.

We have constructed a family of filters \( \{ \mathcal{F}_\alpha \} \) such that \( \mathcal{F}_\alpha \in \sigma(id) \) for
each \( \alpha \in I \) but such that \( \bigwedge_\alpha \mathcal{F}_\alpha \notin \gamma_\alpha(id) \). This shows that \( \kappa \) cannot be a
principal convergence structure.

**Bibliography**

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**Department of Mathematics, St. Lawrence University, Canton, New York**

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