COMMUTATION PROPERTIES OF THE COEFFICIENT MATRIX IN THE DIFFERENTIAL EQUATION OF AN INNER FUNCTION

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Abstract. Let $A(x)$ be an operator valued function that is analytic on the real axis. Assume that $A(x)$ is selfadjoint for each real $x$. It is shown that $A(x)$ and $\int_a^b A(s) \, ds$ commute for all real $x$ iff $A(x)$ and $A(y)$ commute for all real $x$ and $y$. This result is then used to establish several new characterizations of the Potapov inner functions of normal operators $T$ such that $\| T \| < 1$. The case where $\| T \| = 1$, $r(T) < 1$ and $A_T(x)$ and $A_T(y)$ commute for real $x$ and $y$ is discussed. Here $A_T(x) = -i U_T(x) U_T(x)^*$ and $U_T(x)$ is the Potapov inner function for $T$.

This paper is a continuation of the author's study of differentiable inner functions begun in [1], [2], and [4]. It answers some of the questions raised in [1] and improves the characterization of $A_T$ for normal $T$ given in [4]. The results of [4] on the $L^1$ norm of $A_T$ have recently been improved and elaborated on by Helson [7] and Herrero [9].

1. Main result. The main result of this paper is the following theorem. It answers the major unresolved question left in [1]. By operator we mean a bounded linear operator acting in a fixed separable Hilbert space $\mathcal{H}$. If $X$ and $Y$ are operators, then $[X, Y] = XY - YX$.

THEOREM 1. Suppose that $A(x)$ is an operator valued function which is analytic on the real axis. Suppose further that $A(x)$ is selfadjoint for every real $x$. Then the following are equivalent.

(*) $[A(x), A(y)] = 0$ for all real $x$ and $y$.

(**) $[A(x), \int_a^b A(y) \, dy] = 0$ for all real $x$.

To prove this result we need the following lemma.
Lemma 1. If $T_1, T_2$ are selfadjoint operators and $[T_1, [T_1, T_2]]=0$, then $[T_1, T_2]=0$.

Proof. Suppose $T_1$ and $T_2$ are selfadjoint and that $[T_1, [T_1, T_2]]=0$. Then $[T_1, T_2]$ is quasinilpotent [11, p. 4]. But $i[T_1, T_2]$ is selfadjoint since $T_1$ and $T_2$ are. Thus $i[T_1, T_2]=0$ since it is a selfadjoint quasinilpotent operator.

Theorem 1 is written in the notation of [1]. For purposes of a proof, it is notationally more convenient to prove the following result which includes Theorem 1.

Theorem 2. Suppose that $B(x)$ is an operator valued function that is analytic on the real axis. Suppose further that $B(x)$ is selfadjoint for all real $x$. Then the following are equivalent,

(i) $[B'(x), B(x)]=0$ for all real $x$.
(ii) $[B(x), B(y)]=0$ for all real $x$ and $y$.

Proof. Clearly (ii) implies (i). Assume then that (i) holds. Now $B(x)$ admits a series expansion $\sum_{m=0}^{\infty} B_m x^m$ near zero. Repeated differentiations with respect to $x$ show $B_m$ is selfadjoint for every $m$. Following Hellman [5] we insert the series for $B(x)$ into $[B'(x), B(x)]=0$ and get

1. $[B_1, B_0]=0$,
2. $[B_2, B_0]=0$,
3. $3[B_3, B_0]+[B_2, B_1]=0$,
4. $4[B_4, B_0]+2[B_3, B_1]=0$,
5. $5[B_5, B_0]+3[B_4, B_1]+[B_3, B_2]=0$, $\cdots$.

We wish to show that $[B_i, B_j]=0$ for all $i, j \geq 0$. We will say (P) holds for $B_i$ if $[B_i, B_j]=0$ for all $j$. Now $[B_n, B_0]$ is a sum of $[B_i, B_j]$ with $0 \leq i, j \leq n-1$. Thus if $[B_0, B_n]=0$ for $k < n$, then $[B_0, [B_0, B_n]]=0$ and hence $[B_0, B_n]=0$ by Lemma 1. But $[B_1, B_0]=[B_2, B_0]=0$. Thus (P) holds for $B_1$ by induction. But then $[B_2, B_1]=[B_3, B_1]=0$ by (3) and (4). Furthermore $[B_n, B_1]$ is a linear combination of $[B_i, B_j]$ with $1 \leq i, j \leq n-1$. As before we get (P) holds for $B_1$. Continuing in this manner we get that (P) holds for all $B_i$. Thus (ii) holds for all $x$ by analytic continuation.

To see that Theorem 2 implies Theorem 1, let $B(x)=\int A(s) ds$ where $A(s)$ is selfadjoint. By Theorem 2, $[\int A(s) ds, \int A(s) ds]=0$ for all $x$ and $y$ if and only if $[\int A(s) ds, A(s)]=0$ for all $x$. But $[\int A(s) ds, \int A(s) dy]=0$ for all $x$ and $y$ if and only if $[A(s), A(y)]=0$ for all $x$ and $y$. The if part of this last statement is clear. To get the only if, differentiate $[\int A(s) ds, \int A(s) dy]=0$ first with respect to $x$ and then with respect to $y$.

2. Necessity of assumptions. The assumption that $A(x)$ is selfadjoint is necessary. The following example is a modification of one of Hellman's.
Example 1. Let

\[ A(x) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2x \\
4x^2 & 0 & 0 & 0 \\
9x^2 & 0 & 0 & 0
\end{pmatrix}. \]

Then a straightforward calculation verifies \( [A(x), \int_0^x A(y) \, dy] = 0 \). Now the coefficients of the \( x \) and \( x^2 \) terms in the power series expansion of \( A \) do not commute. Hence \( [A(x), A(y)] \) is not always zero.

3. Applications. We review briefly the notation of [3] and [4]. Let \( T \) be an operator such that \( \|T\| \leq 1 \) and the spectral radius, \( r(T) \), is less than one. If \( U_T(w) \) is the Potapov inner function for \( T \), \( |w| \leq 1 \), and \( z = i(1 - w)/(1 + w) \), then let \( U_T(z) = U_T(w) \). The variable \( x \) denotes real values of \( z \). \( U_T(z) \) satisfies the differential equation \( U_T'(x) = iA_T(x)U_T(x) \) where \( A_T(x) \geq 0 \) for all \( x \). A discussion of Potapov inner functions may be found in [10]. The differential equation was introduced in [6] and extensively studied in [3]. The connection between inner functions and operators is nicely explained in [8].

Combining the results of [1], [4], and this paper we get the following theorem.

**Theorem 3.** Suppose that \( \|T\| < 1 \). Let \( U_T(z) \) be the Potapov inner function of \( T \) on the upper half-plane and \( A_T(x) \) be defined by \( U_T'(x) = iA_T(x)U_T(x) \). Then the following are equivalent.

(a) \( T \) is normal.
(b) \( [U_T(x), U_T(y)] = 0 \) for all real \( x \) and \( y \).
(c) \( [A_T(x), A_T(y)] = 0 \) for all real \( x \) and \( y \).
(d) \( [A_T(x), \int_0^x A_T(y) \, dy] = 0 \) for all real \( x \).
(e) \( U_T(x) = \exp(i \int_0^x A_T(y) \, dy) \).
(f) \( [U_T(x), U_T'(x)] = 0 \) for all real \( x \).
(g) \( [A_T(x), U_T(x)] = 0 \) for all real \( x \).
(h) \( [A_T(x), U_T'(x)] = 0 \) for all real \( x \).

**Proof.** The equivalence of (a) and (b) is due to Sherman [12]. The equivalence of (d), (e), (f), (g), and (h) was shown in [1], while (a) and (c) were shown equivalent in [4]. Theorem 1 gives us that (c) and (d) are equivalent.

4. Discussion. It had been initially hoped in [1] that the Potapov inner function for some nonnormal operators would satisfy some type of commutation property and hence be easier to work with. Theorem 3 shows...
that most of the obvious ones are equivalent to the normality of \( T \) if \( \| T \| < 1 \). If \( \| T \| = 1 \) and \( r(T) < 1 \), then (c) does not necessarily imply (a), while (a) always implies (c). An example was given in [4]. That example was \( T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \).

There remains then the possibility that Theorem 1 of [1] could be useful in studying nonspectraloid operators for which (c) holds. It would be of interest, as the next theorem shows, to characterize those operators for which (c) holds.

**Theorem 4.** If \( \| T \| \leq 1, r(T) < 1, \) and \([AT(x), AT(y)]=0\) for all \( x \) and \( y \), then \( T \) has an invariant subspace.

**Proof.** Recall from [1, Theorem 7], that the closure of the range of \( AT(x) \) is independent of \( x \). If there exists an \( x_0 \) such that \( AT(x_0) \) is not a scalar multiple of a projection, then let \( P \) be a nontrivial projection in the spectral resolution of \( AT(x_0) \). Then \([AT(x), P]=0\) for all \( x \) and \( AT(x) = A_1(x) \oplus A_2(x) \). Let \( U_1(x) \) and \( U_2(x) \) be the solutions of \( X' = i A_1 X, X(0) = P \), and \( X' = i A_2 X, X(0) = (I - P) \), respectively. Then \( U_T(x) \) and \( (U_1 \oplus (I - P))(P \oplus U_2)U_T(0) \) both satisfy the initial value problem \( X' = iATX, X(0) = U_T(0) \), and hence are equal. Thus \( U_T \) factors and \( T \) has an invariant subspace. There remains the possibility that \( AT(x) = p(x)P \) for a scalar function \( p(x) \) and projection \( P \). In this case \( U_T(x) = (q(x)P \oplus (I - P))U_T(0) \) where \( q \) is the scalar inner function satisfying \( q'(x) = ip(x)q(x), \; q(0) = 1 \). Clearly \( U_T \) factors unless it is trivial, that is, \( \dim PA = 1 \) and \( q \) is a single Blaschke factor.

The techniques used in proving Theorem 4 can be adapted to prove the following more general result.

**Theorem 5.** Suppose that \( \| T \| \leq 1 \) and \( r(T) < 1 \). Let \( P \) be the projection onto the closure of the range of \( AT(x) \). If there exists an operator \( B \) such that \([B, AT(x)]=0\) for all real \( x \) and \( BP \) is not a scalar multiple of \( P \), then \( T \) has an invariant subspace.

The example mentioned earlier shows that one cannot improve Theorem 5 to produce a reducing subspace since \( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) has none. If we assume \( \| T \| < 1 \), then Theorem 5 can be modified to prove \( T \) has a reducing subspace. See [3, p. 37] for details.

As a first step toward characterizing the nonnormal operators for which (c) holds we show they must be isometric on a nontrivial subspace.

**Theorem 6.** Suppose that \( 1 \geq \| T \| > r(T) \) and that 1 is not an eigenvalue of \( T^*T \). Then \([AT(x), AT(y)]=0\) for all real \( x \) and \( y \) if and only if \( T \) is normal.
Proof. If $T$ is normal, the result follows from the formula for $A_T(x)$ given in [3, p. 31] which expresses $A_T(x)$ in terms of $T$ and $T^*$. Assume then that $1 \geq \|T\| > r(T)$ and 1 is not an eigenvalue of $T^*T$. The case $\|T\| < 1$ was done in [4], so assume $\|T\| = 1$. Let $D = (I - T^*T)^{1/2}$. Then $D$ is a one-to-one selfadjoint operator with a dense range. Up to a scalar function $A_T(x)$ is $D(I - wT)^{-1}(w - T^*)^{-1}D$. Expressing this as a Laurent series convergent for $1 > |w| > r(T)$, we get $\sum_{n=-\infty}^{-1} DBT^{*n-1}Dw^n + \sum_{n=0}^{\infty} D^2T^{*n+1}DBw^n$ where $B = \sum_{m=0}^{\infty} T^mT^*m$. By analytic continuation $[A_T(z), A_T(u)] = 0$ for $z, u$ such that $1 > |(z-i)/(z+i), (u-i)/(u+i)| > r(T)$. Hence all the coefficients in the Laurent series commute. In particular, $DT^iBD^2BD = DBD^2TB$. Hence

$$T^iBD^2 = BD^2T^i \text{ or } [T^i, BD^2] = 0.$$  

Now $$DT^iBD^2BT^*JD = DBT^*JD^2T^iBD.$$  

Hence $$T^iBD^2BT^*JD = BT^*JD^2T^iBD,$$  

and thus $$D^2T^iBT^*JD = T^*JD^2T^iBD \text{ or } [T^*J, D^2T^iBD] = 0.$$  

Setting $i=j=1$ gives $D^2TB = T^*D^2TB$. Multiply both sides by $T^*$ on the right and use the identity $TBT^* = B - I$ to get $D^2(B - I)T^* = T^*D^2(B - I)$. But $D^2BT^* = T^*D^2B$ and hence $[T^*, D^2] = 0$. Thus $T$ and $T^*$ commute with $T^*T$, which in turn implies that $T$ and $T^*$ commute with $B$ also. The earlier identity $D^2TB = T^*D^2TB$ now gives $TT^* = T^*T$, that is, that $T$ is normal.

Theorem 6 improves Theorem 6 of [4] and is probably about the best possible for inner functions analytic on the closed disc. The assumption that 1 is not an eigenvalue of $T^*T$ is equivalent to assuming that $\|T\| < \|\|\|$ for all nonzero $\phi$ in $A$.

References


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