ON THE FIRST COHOMOLOGY GROUP OF DISCRETE GROUPS WITH PROPERTY (T)

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Abstract. Let $G$ be a separable locally compact group with property (T), i.e., the class of one dimensional trivial representations is an isolated point in the dual space $\hat{G}$ of $G$. Let $\pi: G \to O_n$ be a continuous representation of $G$ into the orthogonal group. In this note, we show that $H^1(G, \pi) = 0$.

Let $G$ be a separable locally compact group. Let $\hat{G}$ be the set of all equivalence classes of separable irreducible unitary representations of $G$. We give $\hat{G}$ the inner-hull-kernel topology. $\hat{G}$ is called the dual space of $G$. Following [4], $G$ is said to have property (T) if the equivalence class $I$ of the one dimensional trivial representation of $G$ is an isolated point in $\hat{G}$. Let $\pi: G \to O_n$ be a continuous representation of $G$ into the group of all $n$ by $n$ orthogonal matrices. Let us write $C^1(G, \pi)$ and $B^1(G, \pi)$ for the groups of continuous 1-cocycles and 1-coboundaries respectively. Let $H^1(G, \pi) = C^1(G, \pi)/B^1(G, \pi)$. In this note, we are going to prove that $H^1(G, \pi) = 0$ if $G$ is a separable locally compact group with property (T). Applying to discrete subgroups of $p$-adic groups or real Lie groups, our result generalizes slightly some results in [4] and [3, Theorem 6.5]. Our argument still follows the spirit of [4]. Before we give the proof of the main result, we shall first establish some lemmas needed later.

Lemma 1. Let $H$ be a locally compact group and $\pi: H \to O_n$ a representation. Let $\varphi: H \to R^n$ be a 1-cocycle. If $\varphi(H)$ is bounded in $R^n$, then $\varphi$ is a coboundary.

Proof. Let $K$ be the closure of $\pi(H)$ in $O_n$ and $K \cdot R^n$ the semidirect product of $K$ and $R^n$ where $R^n$ is the normal subgroup and $K$ acts on $R^n$ in the natural manner. Consider then the map $f: H \to K \cdot R^n$ defined by $h \mapsto \pi(h)\varphi(h)$ ($h \in H$). Since $\varphi$ is a 1-cocycle, $f$ clearly is a homomorphism. $\varphi(H)$ is bounded by assumption. It yields that $\text{Cl}(f(H))$ is compact.

Received by the editors January 30, 1973.


Key words and phrases. Locally compact groups, groups with property (T), affine semisimple algebraic groups.

1 Partially supported by NSF Grant GP-29466X.
By the conjugacy theorem, there is \( m \) in \( \mathbb{R}^n \) such that \( m \mathrm{Cl}(f(H))m^{-1} \subset K \). In particular, we have

\[
m \pi(h) \varphi(h)m^{-1} = \pi(h)(m^h \varphi(h)m^{-1}) \in K,
\]

where \( m^h = \pi(h)^{-1}m\pi(h) \) and \( h \in H \). Hence \( m^h \varphi(h)m^{-1} = 1 \) and in additive notation, we get \( \varphi(h) = m - m^h, h \in H \). Therefore \( \varphi \) is a coboundary.

**Lemma 2.** Let \( G \) be a separable locally compact group with property (T). Then there exists a positive number \( \varepsilon \) and a compact subset \( K \) of \( G \) with the following condition: If \( \pi \) is a separable unitary representation of \( G \) on a Hilbert space \( H(\pi) \) and \( x \in H(\pi) \) such that \( \|x\| = 1 \) and \( \|gx, x\| - 1 < \varepsilon \) for all \( g \in K \), then \( \pi \leq I \).

**Proof.** Suppose the assertion to be false. Let \( K_n \) be an increasing sequence of compact subsets of \( G \) such that \( \bigcup_n K_n = G \). Then there are separable unitary representations of \( G, \pi_1, \pi_2, \ldots \) for which there exist \( x_n \in H(\pi_n) \) with \( \|x_n\| = 1 \), \( \|hx_n, x_n\| - 1 < 1/n \) for \( g \in K_n \) and \( \pi_n \not\leq I \) \((n = 1, 2, \ldots)\). Then consider the representation \( \pi = \bigoplus_{n=1}^{\infty} \pi_n \). Due to our construction, \( I \) is contained in the closure of \( \{\pi\} \). However \( G \) has property (T), by [4], [6], \( I \leq \pi \) which implies \( I \leq \pi_n \) for some \( n \). Obviously this is a contradiction.

**Lemma 3.** Let \( G, \varepsilon, K \) be described as in Lemma 2. Let \( \pi \) be a separable unitary representation of \( G \) in the Hilbert space \( H(\pi) \). If \( x \in H(\pi) \) with \( \|x\| = 1 \) and \( \|gx, x\| - 1 < \varepsilon \) for \( g \in K \), then \( \|gx - x\| < 2\sqrt{\varepsilon} \) for all \( g \in G \).

**Proof.** Let us write \( H_1 \) for the set \( \{y \in H(\pi):gy = y \text{ for all } g \in G\} \) and \( H_2 = H_1^\perp \). Clearly, \( H_2 \) is invariant under \( G \), and \( H(\pi) = H_1 \oplus H_2 \). It is also easy to see that \( \pi|_{H_1} \leq I \). Let us write \( x = x_1 + x_2 \) with \( x_1 \in H_1 \) and \( x_2 \in H_2 \). Since \( gx = x_1 + gx_2 \) and \( \|x\| = 1 \),

\[
\|gx - x\| = \|gx_2 - x_2\| < 2\sqrt{\varepsilon} \quad \text{for } g \in K.
\]

Since \( \pi|_{H_2} \not\leq I \) and by Lemma 2, we must have \( \|x_2\| < \varepsilon \). From this, \( \|gx - x\| = \|gx_2 - x_2\| < 2\sqrt{\varepsilon} \), for all \( g \in G \).

**Lemma 4.** Let \( G \) be a separable locally compact group with property (T), \( \pi: G \to O_n \) a continuous homomorphism and \( \varphi: G \to \mathbb{R}^n \) a continuous 1-cocycle. Then \( \varphi(G) \) is bounded in \( \mathbb{R}^n \).

**Proof.** Let \( \varphi: G \to \mathbb{R}^n \) be a continuous 1-cocycle. We define \( \alpha_\lambda: G \to G \cdot \mathbb{R}^n \) (semidirect product) by \( g \mapsto g(\lambda \varphi(g)) \) \((g \in G)\). Clearly the maps

\footnote{Let \( \mathcal{G} \) be the set of all equivalence classes of separable unitary representations of \( G \). We give \( \mathcal{G} \) the inner hull-kernel topology.}
\( \alpha \) (0 \leq \lambda \leq 1) are continuous homomorphisms and, as \( \lambda \to 0 \), \( \alpha_{\lambda} \to \) the inclusion map \( i_{G} \) of \( G \) in \( G \cdot R^{n} \). Now consider the space \( L^{2}(G \cdot R^{n}) = L^{2}(R^{n}) \). Through right translations, \( G \cdot R^{n} \) acts unitarily on \( L^{2}(G \cdot R^{n}) \). Let \( B \) be a unit volume ball in \( R^{n} \) with center at 0, and \( x_{B} \) the characteristic function on \( B \). Since \( B \) is invariant under \( O_{n} \), \( gx_{B} = x_{B} \) for all \( g \in G \). Now let \( \varepsilon < \frac{1}{2} \), and \( G, K \) be described as in Lemma 2. Since \( \alpha_{\lambda} \to i_{G} \) as \( \lambda \to 0 \), there is \( \delta > 0 \) such that

\[ |(\alpha_{\lambda}(g)x_{B}, x_{B}) - 1| < \varepsilon^{2}, \quad g \in K, \]

and \( 0 \leq \lambda \leq \delta \). By Lemma 3, \( \|\alpha_{\lambda}(g)x_{B} - x_{B}\| < 2\sqrt{\varepsilon} \) for all \( g \in G \) and \( 0 \leq \lambda \leq \delta \). By an easy computation, \( \alpha_{\lambda}(g)x_{B} = x_{B_{\lambda}} \), the characteristic function on \( B_{\lambda} \) where \( B_{\lambda} \) is the translation of \( B \) by a vector \( -\lambda\varphi(g)^{\lambda^{-1}} \). Note \( -\lambda\varphi(g)^{\lambda^{-1}} = -\lambda\pi(g^{-1})(\varphi(g)) \) has norm \( \|\lambda\varphi(g)\| \). If \( \|\lambda\varphi(g)\| \geq 1 \), \( B \cap B_{\lambda} = \emptyset \), hence \( \|\alpha_{\lambda}(g)x_{B} - x_{B}\| = \sqrt{2} > 2\sqrt{\varepsilon} \). However

\[ \|\alpha_{\lambda}(g)x_{B} - x_{B}\| < 2\sqrt{\varepsilon} \quad \text{for all} \quad g \in G \]

and \( 0 \leq \lambda \leq \delta \). It follows that \( \|\delta\varphi(g)\| < 1 \) for all \( g \in G \). Hence \( \varphi(G) \) is bounded in \( R^{n} \).

As a consequence of Lemmas 1 and 4, we now have our main result.

**Theorem A.** Let \( G \) be a separable locally compact group with property (T) and \( \pi: G \to O_{n} \) a continuous representation. Then \( H^{1}(G, \pi) = 0 \).

As an application of Theorem A, we have the following vanishing cohomology theorem of certain discrete subgroups.

**Theorem B.** Let \( k \) be a nondiscrete locally compact field of \( \text{ch}(k) = 0 \), \( G \) an affine semisimple algebraic group defined over \( k \) and \( \Gamma \) a discrete subgroup of \( G(k) \). If the \( k \)-rank of each \( k \)-factor of \( G \geq 2 \) and \( G(k)/\Gamma \) has a finite Haar measure, then \( H^{1}(\Gamma, \pi) = 0 \) for every finite dimensional unitary representation \( \pi \) of \( \Gamma \).

**Proof.** By [4], \( \Gamma \) has property (T).

Theorem B generalizes slightly Theorem 6.5 in [3]. However our method cannot be carried out in higher dimensional cohomology groups. In the following, we present a weak rigidity theorem for discrete groups with property (T).

**Theorem C.** Let \( \Gamma \) be a finitely generated discrete group with property (T), \( G \) a compact Lie group, and \( \varphi_{\lambda}: \Gamma \to G \) (0 \leq \lambda \leq 1) a continuous curve of homomorphisms. Then for each \( \lambda \), there is \( g_{\lambda} \in G \) such that \( \varphi_{\lambda}(\gamma) = g_{\lambda}\varphi_{0}(\gamma)g_{\lambda}^{-1} \) for all \( \gamma \in \Gamma \).

**Proof.** Let \( n \) be a positive integer. From [7], we know that there are only finitely many classes of irreducible unitary representations of \( \Gamma \).
with dimension \( \leq n \). Let \( N_j = \text{kernel}(\varphi_j) \). Then the set of kernels \( \{N_j : 0 \leq \lambda \leq 1\} \) is finite. Let us denote this finite set by \( \{M_j, \cdots, M_l\} \). Let \( I_i = \{\lambda : N_\lambda = M_i\} \). We want to show that \( I_i \) (\( i = 1, \cdots, l \)) are closed subsets of \([0, 1]\). Let us write \( \Gamma_i \) for \( \Gamma/M_i \). Clearly if \( \lambda \in \text{Cl}(I_i), N_\lambda \supset M_i \). Hence for each \( \lambda \in \text{Cl}(I_i) \), we set \( \tilde{\varphi}_\lambda \) for the homomorphism of \( \Gamma/M_i \) induced by \( \varphi_\lambda \). Let \( \mathcal{R}(\Gamma_i, G) \) be the space of all representations of \( \Gamma_i \) in \( G \). We equip \( \mathcal{R}(\Gamma_i, G) \) with the compact open topology. \( G \) acts continuously on \( \mathcal{R}(\Gamma_i, G) \) by \( (g \circ \varphi)(\gamma) = g\varphi(\gamma)g^{-1}, \varphi \in \mathcal{R}(\Gamma_i, G), \gamma \in \Gamma_i \) and \( g \in G \). By Theorem A and [8], \( G \circ \tilde{\varphi}_\lambda, (\lambda \in I_i) \) are open subsets of \( \mathcal{R}(\Gamma_i, G) \). Since \( G \) is compact, \( G \circ \tilde{\varphi}_\lambda, (\lambda \in I_i) \) are compact-open in \( \mathcal{R}(\Gamma_i, G) \). On the other hand \( \mathcal{R}(\Gamma_i, G) \) can be viewed as an \( R \)-algebraic variety. It follows that \( \mathcal{R}(\Gamma_i, G) \) has only finitely many arcwise connected components [9]. Hence

\[
\bigcup_{\lambda \in I_i} G \circ \tilde{\varphi}_\lambda = \bigcup_{j=1}^m G \circ \varphi_{\lambda_j}
\]

for some finitely many elements \( \lambda_1, \cdots, \lambda_m \) in \( I_i \), and consequently

\[
\bigcup_{\lambda \in I_i} G \circ \tilde{\varphi}_\lambda \text{ is compact-open in } \mathcal{R}(\Gamma_i, G).
\]

Therefore \( \text{Cl}(I_i) = I_i \) (\( i = 1, \cdots, l \)) and \( l \) has to be 1. Again by Theorem A and [8] \( G \circ \tilde{\varphi}_0 \) is compact-open in \( \mathcal{R}(\Gamma_1, G) \). Since \( \{\tilde{\varphi}_\lambda: 0 \leq \lambda \leq 1\} \) is connected, one concludes readily that \( \tilde{\varphi}_\lambda \in G \circ \varphi_0 \) for all \( \lambda \). Therefore \( \varphi_\lambda \in G \circ \varphi_0 \) for all \( \lambda \).

REFERENCES


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