ON CLOSED SETS OF ORDINALS

HARVEY FRIEDMAN

Abstract. We prove that every stationary set of countable ordinals contains arbitrarily long countable closed subsets.

Call a set $A$ of ordinals closed if and only if every nonempty subset of $A$ which has an upper bound in $A$ has its least upper bound in $A$. It is well known that there are $B \subset \omega_1$ such that neither $B$ nor $\omega_1 - B$ contains an uncountable closed subset. A consequence of what we prove here is that for every $B \subset \omega_1$, either $B$ or $\omega_1 - B$ contains arbitrarily long countable closed subsets.

Call a set $A$ of ordinals $\kappa$-stationary if and only if $A \leq \kappa$ and $A$ intersects every closed subset of $\kappa$ of power $\kappa$. We can restate the above well-known theorem as follows: There is an $A$ such that $A$ and $\omega_1 - A$ are both $\omega_1$-stationary.

We will prove here that every $\omega_1$-stationary set contains arbitrarily long, countable, closed subsets.

Is there a cardinal $\kappa$ such that for all $A \subset \kappa$, either $A$ or $\kappa - A$ contains an uncountable closed subset? Is this true for $\kappa = \omega_2$? Karel Prikry and the author noticed that, in any case, the statement for $\kappa = \omega_2$ cannot be proved true in ZFC.

Theorem. Every $\omega_1$-stationary set contains arbitrarily long countable closed subsets.

Proof. Let $A$ be $\omega_1$-stationary. We prove by induction on $\alpha < \omega_1$ that $A$ has a closed subset of length $\alpha$. Let the induction hypothesis be that
for all $\beta < \alpha$ and for each $\gamma < \omega_1$, there is a closed subset $B \subseteq A$ of length $\beta$, all of whose elements are $> \gamma$.

Case 1. $\alpha$ is a limit ordinal $< \omega_1$. Choose $\beta_0 < \beta_1 < \cdots < \alpha$, with $\sup_n (\beta_n) = \alpha$. Let $\gamma < \omega_1$. By the induction hypothesis, let $B_0 \subseteq A$, $B_0$ of length $\beta_0 + 1$, $B_0$ closed, $(\forall \beta \in B_0) (\beta > \gamma)$. Let $B_{n+1} \subseteq A$, $B_{n+1}$ of length $\beta_{n+1} + 1$, $B_{n+1}$ closed, $(\forall \beta \in B_{n+1}) (\beta > \sup (B_n))$. Then set $B = \bigcup_n B_n$. $B$ has the desired properties.

Case 2. $\alpha = \delta + 2$, $\alpha < \omega_1$. Let $\gamma < \omega_1$. By the induction hypothesis, let $B_0 \subseteq A$ be closed, of length $\delta + 1$, and $(\forall \beta \in B_0) (\beta > \gamma)$. Let $\lambda \in A$ with $\lambda > \sup (B_0)$. Put $B = B_0 \cup \{ \lambda \}$. $B$ has the desired properties.

Case 3. $\alpha = \lambda + 1$, $\lambda$ a limit ordinal $< \omega_1$. Let $\gamma < \omega_1$. By the induction hypothesis, define a sequence of sets $B_0, \xi < \omega_1$, such that

(a) $(\forall \beta \in B_0) (\beta > \gamma)$

(b) if $\xi_1 < \xi_2$ then $(\forall \beta_1 \in B_{\xi_1})(\forall \beta_2 \in B_{\xi_2})(\beta_1 < \beta_2)$

(c) each $B_\xi$ is a closed subset of $A$ of length $\xi$.

Define $f: \omega_1 \rightarrow \omega_1$ by $f(\xi) = \sup (\bigcup_{\xi < \xi} B_\xi)$. Note that the range of $f$ on limit ordinals is an uncountable closed set. Since $A$ is stationary, let $\tau$ be a countable limit ordinal with $f(\tau) \in A$. Choose $\tau_0 < \tau_1 < \cdots < \tau$ with $\sup_n (\tau_n) = \tau$. Let $C_n$ be the first $\lambda_n + 1$ elements of $B_{\tau_n}$. Then set $B^* = \bigcup_n C_n$. $B^*$ is a closed subset of $A$ of length $\lambda$, and $\sup (B^*) = f(\tau) \in A$. Hence $B = B^* \cup \{ f(\tau) \}$ is a closed subset of length at least $\alpha$, all of whose elements are $> \gamma$, and we are done.

The referee has kindly forwarded the following remarks concerning the problems raised on the first page of this paper.

Let us say that a cardinal $K > \omega$ has the property $F$ (briefly, $F(K)$) if for every subset $A$ of $K$ either $A$ or $K - A$ contains a closed subset of order type $\omega_1$.

(1) Silver has shown that the Jensen principle $\square_{\omega_1}$ implies $\neg F(\omega_2)$. Since $\neg \square_{\omega_1}$ is Mahlo in $L$ this gives a lower bound on the proof-theoretic strength of $\text{ZFC} + F(\omega_2)$.

(2) Silver has also observed that in any Cohen extension of any model $M$ of $\text{ZFC}$ obtained by generically collapsing $\omega_1^M$ to $\omega$, $F(K)$ fails for all uncountable $M$-cardinals $K$. (For $A$ take $(x: cf^M(x) = \omega$ and $x < K$).

(3) Solovay has generalized Silver’s proof in (1) above to show that, in $L$, $F(K)$ fails for all cardinals $K > \omega$.

Karel Prikry has informed the author that he has independently shown that $F(K)$ fails for all cardinals $K > \omega$, in $L$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Reference


Department of Mathematics, Stanford University, Stanford, California 94305

Department of Mathematics, State University of New York at Buffalo, Amherst, New York 14226