ON CLOSED SETS OF ORDINALS

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Abstract. We prove that every stationary set of countable ordinals contains arbitrarily long countable closed subsets.

Call a set $A$ of ordinals closed if and only if every nonempty subset of $A$ which has an upper bound in $A$ has its least upper bound in $A$. It is well known that there are $B \subseteq \omega_1$ such that neither $B$ nor $\omega_1 - B$ contains an uncountable closed subset. A consequence of what we prove here is that for every $B \subseteq \omega_1$, either $B$ or $\omega_1 - B$ contains arbitrarily long countable closed subsets.

Call a set $A$ of ordinals $\kappa$-stationary if and only if $A \subseteq \kappa$ and $A$ intersects every closed subset of $\kappa$ of power $\kappa$. We can restate the above well-known theorem as follows: There is an $A$ such that $A$ and $\omega_1 - A$ are both $\omega_1$-stationary.

We will prove here that every $\omega_1$-stationary set contains arbitrarily long, countable, closed subsets.

Is there a cardinal $\kappa$ such that for all $A \subseteq \kappa$, either $A$ or $\kappa - A$ contains an uncountable closed subset? Is this true for $\kappa = \omega_2$? Karel Prikry and the author noticed that, in any case, the statement for $\kappa = \omega_2$ cannot be proved true in ZFC.

Theorem. Every $\omega_1$-stationary set contains arbitrarily long countable closed subsets.

Proof. Let $A$ be $\omega_1$-stationary. We prove by induction on $\alpha < \omega_1$ that $A$ has a closed subset of length $\alpha$. Let the induction hypothesis be that

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The Theorem was obtained in 1968.

2 In fact, Solovay [1] proves that for uncountable regular cardinals $\kappa$, every $\kappa$-stationary set is the union of $\kappa$ disjoint $\kappa$-stationary sets.

3 By adding an $f: \omega_2 \rightarrow \{0, 1\}$ generic with respect to the partial ordering of countable partial $g: \omega_2 \rightarrow \{0, 1\}$. If the ground model satisfies $\text{ZFC} + 2^{\omega_0} = \omega_1$, then in the forcing extension cardinals are preserved, $\{\alpha: f(\alpha) = 1\}$ and $\{\alpha: f(\alpha) = 0\}$ contain no uncountable closed subsets, and $2^{\omega_0} = \omega_1$ holds.

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for all \( \beta < \alpha \) and for each \( \gamma < \omega_1 \), there is a closed subset \( B \subseteq A \) of length \( \beta \), all of whose elements are \( > \gamma \).

**Case 1.** \( \alpha \) is a limit ordinal \( < \omega_1 \). Choose \( \beta_0 < \beta_1 < \cdots < \alpha \), with \( \sup_n (\beta_n) = \alpha \). Let \( \gamma < \omega_1 \). By the induction hypothesis, let \( B_0 \subseteq A \), \( B_0 \) of length \( \beta_0 + 1 \), \( B_0 \) closed, \( (\forall \beta \in B_0) (\beta > \gamma) \). Let \( B_{n+1} \subseteq A \), \( B_{n+1} \) of length \( \beta_{n+1} + 1 \), \( B_{n+1} \) closed, \( (\forall \beta \in B_{n+1}) (\beta > \sup(B_n)) \). Then set \( B = \bigcup_n B_n \). \( B \) has the desired properties.

**Case 2.** \( \alpha = \delta + 2 \), \( \alpha < \omega_1 \). Let \( \gamma < \omega_1 \). By the induction hypothesis, let \( B_0 \subseteq A \) be closed, of length \( \delta + 1 \), and \( (\forall \beta \in B_0) (\beta > \gamma) \). Let \( \lambda \in A \) with \( \lambda > \sup(B_0) \). Put \( B = B_0 \cup \{ \lambda \} \). \( B \) has the desired properties.

**Case 3.** \( \alpha = \lambda + 1 \), \( \lambda \) a limit ordinal \( < \omega_1 \). Let \( \gamma < \omega_1 \). By the induction hypothesis, define a sequence of sets \( B_1, \xi < \omega_1 \), such that

1. \( (\forall \beta \in B_0) (\beta > \gamma) \)
2. if \( \xi_1 < \xi_2 \) then \( (\forall \beta_1 \in B_{\xi_1})(\forall \beta_2 \in B_{\xi_2})(\beta_1 < \beta_2) \)
3. each \( B_\xi \) is a closed subset of \( A \) of length \( \lambda \).

Define \( f : \omega_1 \to \omega_1 \) by \( f(\xi) = \sup(\bigcup_{\xi < \xi} B_{\xi}) \). Note that the range of \( f \) on limit ordinals is an uncountable closed set. Since \( A \) is stationary, let \( \tau \) be a countable limit ordinal with \( f(\tau) \in A \). Choose \( \tau_0 < \tau_1 < \cdots < \tau \) with \( \sup_n (\tau_n) = \tau \). Let \( C_n \) be the first \( \lambda_n + 1 \) elements of \( B_{\tau_n} \). Then set \( B^* = \bigcup_n C_n \). \( B^* \) is a closed subset of \( A \) of length \( \lambda \), and \( \sup(B^*) = f(\tau) \in A \). Hence \( B = B^* \cup \{ f(\tau) \} \) is a closed subset of length at least \( \alpha \), all of whose elements are \( > \gamma \), and we are done.

The referee has kindly forwarded the following remarks concerning the problems raised on the first page of this paper.

Let us say that a cardinal \( K > \omega \) has the property \( F \) (briefly, \( F(K) \)) if for every subset \( A \) of \( K \) either \( A \) or \( K \setminus A \) contains a closed subset of order type \( \omega_1 \).

1. Silver has shown that the Jensen principle \( \Box_{\omega_1} \) implies \( \neg F(\omega_2) \). Since \( \neg \Box_{\omega_1} \to \omega_2 \) is Mahlo in \( L \) this gives a lower bound on the proof-theoretic strength of \( ZFC + F(\omega_2) \).

2. Silver has also observed that in any Cohen extension of any model \( M \) of \( ZFC \) obtained by generically collapsing \( \omega_1^M \) to \( \omega \), \( F(K) \) fails for all uncountable \( M \)-cardinals \( K \). (For \( A \) take \( \{ \alpha : cf^M(\alpha) = \omega \) and \( \alpha < K \} \).)

3. Solovay has generalized Silver's proof in (1) above to show that, in \( L \), \( F(K) \) fails for all cardinals \( K > \omega \).

Karel Prikry has informed the author that he has independently shown that \( F(K) \) fails for all cardinals \( K > \omega \), in \( L \).

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