STEINITZ CLASSES IN QUARTIC FIELDS

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Abstract. Let $K$ be normal quartic over the rationals. Let $l \equiv 3 \pmod{4}$ be an odd prime. If the class number of $K$ is even, there is a normal extension $L$ of degree $l$ over $K$ such that the relative discriminant is principal, but $L$ has no relative integral base over $K$.

I. Introduction and results. Let $K$ be an algebraic number field, and $L$ a finite extension. The relative discriminant $D_{L/K}$ is an ideal of $K$. Let $d$ be the discriminant of a $K$-base of $L$ and $(d)$ the principal ideal generated by $d$. Then $D_{L/K} = B^2(d)$ for some fractional ideal $B$ of $K$. The ideal class to which $B$ belongs is written $C(L/K)$ and is called the Steinitz class of $L$ with respect to $K$.

Artin [1] showed that $L$ has a relative integral base over $K$ if and only if $C(L/K)$ is principal. Thus if the class number $h_K$ is odd, $L$ has a relative integral base if and only if $D_{L/K}$ is principal.

The story is different if $h_K$ is even; $C(L/K)$ may be in a class of order 2, i.e., $D_{L/K}$ can be principal without $L$ having an integral $K$-base.

Fröhlich [2] showed that every ideal class of $K$ is a Steinitz class for some quadratic extension. For a fixed odd prime $l$, Long [5] found which classes of $K$ can be Steinitz classes for some normal extension of degree $l$. We repeat his result. The classes are those of the form $C^{l-1/2}$, where $C$ is a class containing a prime divisor of $l$ or $C$ contains a prime which splits fully upon adjunction of the $l$th roots of unity.

Let $K$ be an algebraic number field, and let $l$ be an odd prime. We say $K$ has property $(\ast)$ with respect to $l$ if there is a normal extension $L$ of degree $l$ which has no relative integral base, but $D_{L/K}$ is principal.

No field $K$ with odd class number can have $(\ast)$ with respect to any prime; $D_{L/K}$ is principal if and only if $L$ has a relative integral base. Thus, for the rest of the paper, we only deal with fields $K$ for which $h_K$ is even.

If $l \equiv 1 \pmod{4}$ and $h_K = 2$, it is clear that $K$ does not have $(\ast)$ with respect to $l$. For the case $l \equiv 3 \pmod{4}$ and $h_K$ even, the problem seems harder. We do not know of any such fields which do not have property $(\ast)$ with respect to $l$.

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Theorem 1. Let $K$ be quadratic over the rationals $Q$. Suppose $h_K$ is even and $l \equiv 3 \pmod{4}$ is prime. Then $K$ has $(\ast)$ with respect to $l$.

Theorem 2. Let $K$ be normal quartic over $Q$. Suppose $h_K$ is even and $l \equiv 3 \pmod{4}$ is prime. Then $K$ has $(\ast)$ with respect to $l$.

II Proofs. First, some preliminary remarks. The ideal classes of $K$ which are Steinitz classes for some normal extension of $K$ of degree $l$ form a group [5]. If $K$ does not have $(\ast)$ then all primes which split fully upon adjunction of an $l$th root of unity $\zeta$ are in classes of odd order. Thus the 2-part of the Hilbert class field of $K$ lies in $K(\zeta)$ and hence $h_K \equiv 2 \pmod{4}$. Also $K((-l)^{1/2})$ is quadratic unramified over $K$ and $K$ is totally imaginary.

Theorem 1 is easy to complete. We have $\zeta$ imaginary and $l \mid D_{K/Q}$. Now $K \not= Q((-l)^{1/2})$, since $-l$ is not a square in $K$; thus $l$ is the square of a prime ideal in a class of order 2.

We divide Theorem 2 into two cases. First assume $K$ is cyclic over $Q$. Let $k$ be the unique subfield; $k$ is real. A prime fully ramified from $Q$ to $K$ is $\equiv 1 \pmod{4}$ or is 2. Any prime ramified in $k$ is fully ramified in $K$. Thus $l$ is ramified from $k$ to $K$.

Let $h_0$ be the narrow class number of $k$. Let $t$ be the number of primes (including infinite primes) ramifying from $k$ to $K$. By a formula of Hasse [3, p. 99], the number $h$ of ambiguous classes of $K$ over $k$ is

$$(1) \quad h = h_0 2^{t+q^*-3}$$

and the number $h'$ of ambiguous classes of $K$ containing ambiguous ideals is

$$(2) \quad h' = h_0 2^{t+q^*-3}$$

where $q^*$, $q$ are given by

$$(3) \quad 2q^* = (E_k \cap N_{K/k}K^*:E_k^2),$$

$$ (4) \quad 2q = (E_k \cap N_{K/k}E_K:E_k^2).$$

In (3), (4), $E_K$, $E_k$ are the unit groups.

The ambiguous classes of $K$ are a group, and since $h_K \equiv 2 \pmod{4}$, we also have $h \equiv 2 \pmod{4}$. In the case of $K$ cyclic over $Q$, we have $t \geq 4$ and hence $h_0$ is odd and $q^* = 0$. Thus $k = Q(p^{1/2})$ or $Q(2^{1/2})$ and $p \equiv 1 \pmod{4}$. Now $l$ is inert in $k$; otherwise $t \geq 5$. Thus $D_{K/k} = (lp^{1/2})$. It follows that $l$ is the square of a prime in the class of order 2 in $K$.

Next, let $K$ have Galois group $C_2 \times C_2$. Let $k$ be the real subfield. Suppose $l$ ramifies from $Q$ to $k$. Then $2 \mid h_0$ and the fundamental unit $\epsilon$ of $k$ has norm 1. Hasse's formula yields $t = 3$, $q^* = 0$; $t = 2$, $q^* = 1$; or $t = 2$, $q^* = 2$. 

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q* = 0. Since \( \varepsilon \) is totally positive, \( \varepsilon \) is a norm at all primes except possibly one; hence \( \varepsilon \) is a global norm and \( q^* \geq 1 \). Our only alternative is \( t = 2, q^* = 1, h_0 \equiv 2 \) (4). Then \( K \) must be \( k((-l)^{1/2}) \) which contradicts the fact that \( -l \) is not a square in \( K \).

Finally, suppose \( l \) does not ramify in \( k \). In (1), \( t \geq 3 \) and our alternatives are \( t = 3, q^* = 0; t = 3, q^* = 1; t = 4, q^* = 0 \). If \( q^* = 1, t = 3 \), then \( h_0 \) is odd and \( k = Q(p^{1/2}) \), \( p \equiv 1 \) (4) a prime or \( p = 2 \). In either case, \( \varepsilon \) is not totally positive, so \( q^* \neq 1 \).

In the other cases, \( q^* = 0, l \). If \( t = 3, l \) is the only finite prime ramifying, so \( l \) ramifies in the class of order 2. If \( t = 4, h_0 \) is odd and \( k = Q(p^{1/2}) \) or \( Q(2^{1/2}) \) as before. Then \( k' = Q((-l)^{1/2}) \) is a subfield of \( K \), where \( r \) is a prime different from \( p \). Thus \( l, r \) are inert in \( k \) and ramify from \( k \) to \( k' \). Hence the prime in \( K \) dividing \( l \) must lie in the class of order 2.

III. Additional remarks. It is clear that any normal extension \( K \) of \( Q \) with even class number must have property (*) with respect to any odd prime \( l \), simply because \( l \) cannot have even ramification index in \( K \).

If \( l \equiv 3 \) (4) and \( K \) is an abelian field with even class number and not having property (*), then the largest subfield of \( K \) which has degree a power of 2 also does not have (*).

REFERENCES


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