ON THE RANGE OF A HOMOMORPHISM OF A GROUP ALGEBRA INTO A MEASURE ALGEBRA

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Abstract. It is shown, that if $G$ is a LCA group and if $H$ is a nondiscrete LCA group then there exists a proper closed subalgebra of the measure algebra of $H$ (independent of the choice of $G$) in which the range of every homomorphism of the group algebra of $G$ into the measure algebra of $H$ is contained.

Throughout this paper, $G$ and $H$ denote LCA groups and $\hat{G}$ and $\hat{H}$ denote their dual groups, respectively. $\mathcal{X}(H)$ is the set of all the locally compact group topologies of $H$ which are at least as strong as the original one of $H$. For each $\tau \in \mathcal{X}(H)$, if we denote by $H^{'\tau}$ a LCA group with underlying group $H$ and topology $\tau$, the natural continuous isomorphism of $H^{'\tau}$ onto $H$, $x \in H^{'\tau} \mapsto x \in H$, induces a natural norm-preserving embedding of $L^1(H^{'\tau})$ into $M(H)$, which we also denote by $L^1(H^{'\tau})$. For the other notations and terminologies which we need in this paper, we follow [6].

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Theorem. If $h$ is a homomorphism of $L^1(G)$ into $M(H)$, then there exist finitely many elements $\tau_1, \tau_2, \cdots, \tau_n \in \mathcal{X}(H)$ such that the range of $h$ is contained in $\sum_{i=1}^n L^1(H^{'\tau_i})$.

For the proof of the theorem, we essentially use Cohen's results, which determine all the homomorphisms of $L^1(G)$ into $M(H)$ by the notion of the coset ring and piecewise affine maps (cf. [1], [2], [3] and [6, Chapters 3 and 4]).

If $h$ is a homomorphism of $L^1(G)$ into $M(H)$, Cohen's theorem asserts that there exist $Y$, an element of the coset ring of $\hat{H}$, and a piecewise affine map $\alpha$ from $Y$ into $\hat{G}$ such that

$$h(f)^{(\alpha)}(r) = f(\alpha(r)), \quad r \in Y$$

$$= 0, \quad r \notin Y \quad (f \in L^1(G)),$$

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and conversely, if $Y$ is an element of the coset ring of $\hat{H}$ and if $\alpha$ is a piecewise affine map from $Y$ into $\hat{G}$, the pair $(Y, \alpha)$ induces a unique homomorphism $h$ of $L^1(G)$ into $M(H)$ which satisfies (1). We call the pair $(Y, \alpha)$ the dual map of $h$ after P. Eymard [3] (though slightly different from his definition).

For the rest of this paper, $h$ denotes a homomorphism of $L^1(G)$ into $M(H)$ and $(Y, \alpha)$ denotes the dual map of $h$.

**Lemma 1.** If $Y$ is an open subgroup of $\hat{H}$ and $\alpha$ is a continuous homomorphism from $Y$ into $\hat{G}$, then the range of $h$ is contained in $L^1(H^\tau)$ for some $\tau \in \mathcal{I}(H)$.

**Proof.** We suppose first that $Y=\hat{H}$ and $\alpha(Y)$ is dense in $\hat{G}$, and then there exists a natural continuous isomorphism $\hat{\alpha}$ of $G$ into $H$ such that

$$ (\hat{\alpha}(x), r) = (x, \alpha(r)) \quad (x \in G, r \in \hat{H}). $$

We can introduce in $H$ a locally compact group topology $\tau$ such that $\hat{\alpha}$ becomes an open continuous map of $G$ into $H^\tau$, and then $\hat{\alpha}$ induces the natural isomorphism of $L^1(G)$ into $L^1(H^\tau)$, which just coincides with $h$.

Next we suppose only that $\alpha(Y)$ is dense in $\hat{G}$. By the above considerations, we have an element $\tau \in \mathcal{I}(H/L)$ and a continuous isomorphism $h$ of $L^1(G)$ into $L^1((H/L)^\tau)$ such that the dual map of $h$ is $(Y, \alpha)$, where $L$ denotes the annihilator of $Y$ in $H$.

Let $\pi$ be the natural map of $H$ onto $H/L$. If we introduce in $H$ a topology $\tau$ with a basis $\{\pi^{-1}(V) \cap W : V$ is open in $(H/L)^\tau$ and $W$ is open in $H\}$, then $\tau$ is a locally compact group topology of $H$, and the map $\pi$ induces an open continuous homomorphism of $H^\tau$ onto $(H/L)^\tau$. For each $f \in L^1((H/L)^\tau)$, put $h'(f) = f \circ \pi$, then $h'(f)$ belongs to $L^1(H^\tau)$ and $h'h= h$. Thus we have $h(L^1(G)) = h'h(L^1(G)) \subset L^1(H^\tau)$.

Finally we prove the general case. Let $\Lambda$ be the closure of $\alpha(Y)$ in $\hat{G}$. Then there exists a homomorphism $h''$ of $L^1(G/K)$ into $M(H)$ with the dual map $(Y, \alpha)$, where $K$ is the annihilator of $\Lambda$ in $G$. Since $A(\Lambda)$ coincides with the set $\{f|_\Lambda : f \in A(\hat{G})\}$, we can reduce the problem to the preceding case; thus we have $h(L^1(G)) = h''(L^1(G/K)) \subset L^1(H^\tau)$ for some $\tau \in \mathcal{I}(H)$. This completes the proof.

**Lemma 2.** If $Y$ is an open coset and $\alpha$ is an affine map, then we get the same conclusion as Lemma 1.

**Proof.** Let $r_2$ be an element of $H$ such that $Y-r_2$ is an open subgroup of $\hat{H}$. There exist a continuous homomorphism $\beta$ of $Y-r_2$ into $\hat{G}$ and $r_1 \in \hat{G}$ such that

$$ \alpha(r) = \beta(r - r_2) - r_1 \quad (r \in Y). $$
By Lemma 1, there exist an element \( r \in \mathcal{X}(H) \) and a continuous homomorphism \( h' \) of \( L^1(G) \) into \( L^1(H') \) with the dual map \((Y \rightarrow r_2, \beta)\). If we define \( h_1 \) and \( h_2 \) by

\[
    h_1(f) = r_1 f \quad (f \in L^1(G)); \quad h_2(g) = r_2 g \quad (g \in L^1(H'))
\]

then \( h_1 \) and \( h_2 \) are homomorphisms of \( L^1(G) \) into \( L^1(H) \) and \( L^1(H') \) into \( L^1(H') \), respectively. Since \( h = h_2 h' h_1 \), the range of \( h \) is contained in \( L^1(H') \) and Lemma 2 is proved.

Let \( J(H) \) be the set of all the idempotent measures in \( M(H) \), and for each \( \mu \in J(H) \) we put \( S(\mu) = [r \in \hat{H} : \hat{\mu}(r) = 1] \).

**Lemma 3.** If \( \mu \) is an element of \( J(H) \), then there exist finitely many compact subgroups \( K_1, K_2, \ldots, K_n \) of \( H \) such that

(i) \( m_{K_i} \) and \( m_{K_j} \) are mutually singular for \( i \neq j \),

(ii) for \( i \) and \( j \), we have \( m_{K_i} * m_{K_j} = m_{K_i + K_j} \ll m_{K_i} \) (absolutely continuous with respect to \( m_{K_i} \)) for some \( l \),

(iii) \( \mu \ll \sum_{i=1}^{n} m_{K_i} \),

where \( m_{K} \) denotes the Haar measure of a compact group \( K \).

**Proof.** There exists a set \([K_1, K_2, \ldots, K_m]\) of finitely many compact subgroups of \( H \) which satisfies the conditions (i) and (ii) (cf. [5]). We can choose finitely many compact subgroups \( K_{m+1}, \ldots, K_n \) of \( H \) (if necessary) so that \([K_1, K_2, \ldots, K_n]\) satisfies the conditions (i), (ii) and (iii), and this completes the proof.

**Lemma 4.** If there exist an open coset \( \Lambda \) and an affine map \( \tilde{\alpha} \) of \( \Lambda \) into \( \hat{G} \) such that \( Y \subset \Lambda \), \( \tilde{\alpha}|_Y = \alpha \), then we get the conclusion of the theorem.

**Proof.** Since \( Y \) is an element of the coset ring of \( \hat{H} \), there exists \( \mu \in J(H) \) such that \( S(\mu) = Y \). Since \( \mu \) is determined by \( h \), we express \( \mu \) by \( j(h) \). Let \([K_1, K_2, \ldots, K_n]\) be a set of finitely many compact subgroups of \( H \) which satisfies (i), (ii) and (iii) of Lemma 3. We decompose \( \mu \) as

\[
    \mu = \lambda_1 + \lambda_2 + \cdots + \lambda_n \quad (i=1, 2, \ldots, n),
\]

and we proceed by induction on the number \( n \) of \([K_1, K_2, \ldots, K_n]\). Thus we suppose that Lemma 4 is true if \( n \leq k \), and prove that Lemma 4 is also true for \( n = k + 1 \).

We can suppose without loss of generality that \( K_n \) is minimal in the sense that \( K_i / K_n \cap K_i \) is infinite for \( i \neq n \). Then since \( \mu = \mu * \mu = \lambda_n * \lambda_n + \sum_{i \neq n} \lambda_i * \lambda_i \), we get \( \sum_{i \neq n} \lambda_i * \lambda_i \ll \sum_{i=1}^{n} m_{K_i} \) and \( \lambda_n \in J(H) \).

If we put

\[
    h_1: f \in L^1(G) \mapsto h(f) * \lambda_n * \mu \in M(H),
    h_2: f \in L^1(G) \mapsto h(f) * (\mu - \mu * \lambda_n) \in M(H),
\]

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then $h_1$ and $h_2$ are homomorphisms which satisfy $h_1(f)+h_2(f)=h(f)$ ($f \in L^1(G)$). Since $[K_1, K_2, \ldots, K_{n-1}]$ satisfies the conditions (i), (ii) and (iii) of Lemma 3 for $\mu=j(h_2)$, we have by the assumption of the induction that $h_2(L^1(G)) \subseteq \sum_{r \in A} L^1(H')$ for some finite subset $A \subset \mathcal{I}(H)$, and we have only to prove the lemma for $h=h_1$. Therefore we can assume here without loss of generality that $\lambda_n * \mu = \mu$, that is $S(\lambda_n) \supseteq S(\mu)$. Obviously, $\lambda_n$ is an irreducible idempotent, and hence there exist $r_1, r_2, \ldots, r_m \in \hat{H}$ such that $d\lambda_n = [(x, r_1) + \cdots + (x, r_m)] dm_{\lambda_n}$, where $r_i - r_j$ $(i \neq j)$ does not belong to the annihilator of $\lambda_n$.

For each $i$, let $\sigma_i$ be an element of $J(H)$ such that $d\sigma_i = (x, r_i) dm_{\lambda_n}$ and let $h_i$ be a homomorphism of $L^1(G)$ into $M(H)$ with the dual map $(S(\sigma_i) \cap \Lambda, \bar{a}|_{S(\sigma_i) \cap \Lambda})$. Let $h'_i$ and $h''_i$ be homomorphisms of $L^1(G)$ into $M(H)$ such that $h'_i(f) = h(f) * \sigma_i$ and $h''_i(f) = h(f) * (\sigma_i - \sigma_i * \mu)$, and then we have $h'_i(f) = h_i(f) - h''_i(f)$ ($f \in L^1(G)$). By Lemma 2, $h_i$ maps $L^1(G)$ into $L^1(H')$ for some $\tau_i \in \mathcal{I}(H)$, and since $j(h''_i)$ is absolutely continuous with respect to $\sum_{i=1}^{n-1} m_{\lambda_i}$, we have again by the assumption of the induction that $h''_i$ maps $L^1(G)$ into $\sum_{r \in B_i} L^1(H')$ for some finite subset $B_i \subset \mathcal{I}(H)$, and consequently we get

\[ h(L^1(G)) \subseteq \sum_{i=1}^{m} h_i(L^1(G)) - \sum_{i=1}^{m} h''_i(L^1(G)) \subseteq \sum_{i=1}^{m} L^1(H'), \]

and this completes the proof.

The proof of the theorem. Let $(Y, \alpha)$ be the dual map of $h$. There exist a set of pairwise disjoint elements $\{Y_i\}_{i=1}^n$ of the coset ring of $\hat{H}$, a set of open cosets $\{K_i\}_{i=1}^n$ of $\hat{H}$ and a set of affine maps $\{\alpha_i : K_i \to \hat{G}\}_{i=1}^n$ such that

\[ Y = Y_1 \cup \cdots \cup Y_n, \quad K_i \supseteq Y_i, \quad \alpha|_{Y_i} = \alpha_i|_{Y_i} \quad (i = 1, 2, \ldots, n). \]

If we denote by $h_i$ a homomorphism of $L^1(G)$ into $M(H)$ with the dual map $(Y_i, \alpha|_{Y_i})$ $(i = 1, 2, \ldots, n)$, then we have $h(f) = h_1(f) + \cdots + h_n(f)$ ($f \in L^1(G)$). By Lemma 4 we have $h_i(f) \in \sum_{r \in A_i} L^1(H')$ for some finite subset $A_i \subset \mathcal{I}(H)$, and hence $h(f)$ belongs to $\sum_{r \in A_i} L^1(H')$ for each $f \in L^1(G)$ and $i$, and thus the theorem is proved.

Remark. If we refer to [4], we can see that $\sum_{r \in \mathcal{I}(H)} L^1(H')$ is a subalgebra of $M(H)$ and that the norm closure of $\sum_{r \in \mathcal{I}(H)} L^1(H')$ in $M(H)$ is a proper closed subalgebra of $M(H)$ if $H$ is not discrete. This means that the set of the elements of the form $h(x)$ ($x \in G$), where a LCA group $G$ and a homomorphism $h$ of $L^1(G)$ into $M(H)$ vary arbitrarily, constitutes the subalgebra $\sum_{r \in \mathcal{I}(H)} L^1(H')$ contained (if $H$ is not discrete) in a proper closed subalgebra of $M(H)$.
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