A SUFFICIENT CONDITION FOR NONVANISHING OF DETERMINANTS

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Abstract. In this note we derive sufficient conditions for a diagonally dominant reducible matrix to be nonsingular.

1. Throughout this note we are concerned with $A = (a_{ij})$, an $n \times n$ matrix which is diagonally dominant and where

$$J = \left\{ i \in N \left| |a_{ii}| > \sum_{j=1; j \neq i}^{n} |a_{ij}| \right. \right\} \neq \emptyset$$

where $N = \{1, 2, \ldots, n\}$. If $J = N$, $A$ is strictly diagonally dominant and then the Gersgorin circle theorem implies that the determinant of $A$ does not vanish [1, p. 106]. If $A$ is irreducible, Taussky [5] has shown that $A$ is nonsingular. In this note, we prove the following theorem.

**Theorem.** Let the matrix $A$ be such that for each $i \notin J$ there is a sequence of nonzero elements of $A$ of the form $a_{i_1 i_2}, a_{i_1 i_3}, \ldots, a_{i_1 r}$ with $j \in J$. Then $A$ is nonsingular.

2. We need the following lemma and results to prove the Theorem.

**Lemma 1.** Let $A$ satisfy the conditions of the theorem. Then for any nonempty subset $L$ of $N$ such that $L \cap J = \emptyset$, there is a nonzero element $a_{ij}$ with $i \in L$ and $j \notin L$.

**Proof.** Let $L$ be a nonempty subset of $N$ such that $L \cap J = \emptyset$. Choose $i_1 \in L$, then $i_1 \notin J$ and, hence, there is a sequence of nonzero elements of $A$ of the form $a_{i_1 i_2}, a_{i_1 i_3}, \ldots, a_{i_{r-1} i_r}$ for some $i_r \in J$. Let $r$ be the first integer such that $i_r \notin L$ and note that $2 \leq r \leq s$ since $i_1 \in L$ and $i_s \notin L$. Then $a_{i_r i_{r-1}} \neq 0$ with $i_{r-1} \in L$ and $i_r \notin L$. This proves Lemma 1.

**Corollary 2.** Let $A$ satisfy the conditions of the Theorem. If $J = \{i_1, i_2, \ldots, i_n\}$, then there is a permutation $(i_1 i_2 \cdots i_n)$ of $N$ such that, for each $j = k + 1, \ldots, n$, $a_{i_l i_j} \neq 0$ for some $l < j$. 

Received by the editors February 22, 1973.


Key words and phrases. Matrix, determinant, diagonally dominant, reducible, irreducible, nonsingular, $M$-matrix.

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Proof. If \( J = N \), then there is nothing to prove. Suppose \( J \neq N \), then \( L_1 = N - J \) is nonempty and \( L_1 \cap J = \emptyset \). Hence, by Lemma 1, there are an \( i_{k+1} \in L_1 \) and \( j \notin L_1 \) such that \( a_{i_{k+1}j} \neq 0 \). Since \( j \notin L_1 \), we have \( j \in N - L_1 = J \). Hence \( j = i_l \) for some \( 1 \leq l \leq k \).

Let \( L_2 = L_1 - \{i_{k+1}\} \). If \( L_2 = \emptyset \), then the proof is completed. Suppose \( L_2 \neq \emptyset \). Obviously \( L_2 \cap J = \emptyset \). Again, by Lemma 1, there are \( i_{k+2} \in L_2 \) and \( j \notin L_2 \) such that \( a_{i_{k+2}j} \neq 0 \). Since \( j \notin L_2 \), we have \( j \in J \cap \{i_{k+1}\} \). Hence \( j = i_l \) for some \( 1 \leq l \leq k + 1 \). The corollary is proved by repeating the above process until \( L_p = L_{p-1} - \{i_{k+p-1}\} = \emptyset \).

Corollary 3. Let \( A \) satisfy the conditions of the Theorem. If \( J = \{i_1, i_2, \ldots, i_k\} \), then there is a permutation \( (i_1, i_2, \ldots, i_n) \) of \( N \) such that

\[
g_j = |a_{ij}| - \sum_{l=j+1}^{n} |a_{il}|, \quad j = 1, 2, \ldots, n,
\]

are positive.

Proof. From Corollary 2, there is a permutation \( (i_1, i_2, \ldots, i_n) \) of \( N \) such that, for each \( j = k+1, \ldots, n \), \( a_{ij} \neq 0 \) for some \( l < j \). Notice that if \( i_j \in J \), then

\[
g_j = |a_{ij}| - \sum_{l=j+1}^{n} |a_{il}| \geq |a_{ij}| - \sum_{l=j+1}^{n} |a_{il}| > 0.
\]

Hence \( g_j > 0 \), \( j = 1, 2, \ldots, k \). For \( j = k+1, \ldots, n \),

\[
g_j = |a_{ij}| - \sum_{l=j+1}^{n} |a_{il}| > |a_{ij}| - \sum_{l=j+1}^{n} |a_{il}| - \sum_{l=1}^{j-1} |a_{il}| \geq 0
\]

since \( a_{ij} \neq 0 \) for some \( l < j \). This completes the proof.

Corollary 4. Let \( A \) satisfy the conditions of the Theorem. If \( A \) is real and \( a_{ij} \leq 0 \), \( a_{jj} > 0 \), then \( A \) is an M-matrix [2].

Proof. Let \( J = \{i_1, i_2, \ldots, i_k\} \). Then it follows from Corollary 3 that there is a permutation \( (i_1, i_2, \ldots, i_n) \) of \( N \) such that

\[
g_j = |a_{ij}| - \sum_{l=j+1}^{n} |a_{il}| > 0, \quad j = 1, 2, \ldots, n.
\]

Now, \( \det A = g_1 \det A(i_1) + \det A(i_j) \), where \( A(i, j, \ldots, k) \) denotes the matrix obtained from \( A \) with \( i-, j-, \ldots, k \)-rows and columns deleted, and \( A(i_j) \) is the matrix \( A \) with the entry \( a_{ij} \) replaced by \( \sum_{l=j+1}^{n} a_{il} \). From [1, p. 294, Problem 9], we have \( \det A(i_j) \geq 0 \). Thus

\[
\det A \geq g_1 \det A(i_1).
\]
Similarly, we have
\[ \det A(i_1) \geq g_2 \det A(i_1, i_2), \]
\[ \vdots \]
\[ \vdots \]
\[ \det A(i_1, i_2, \ldots, i_{n-1}) \geq g_n. \]

Hence \( \det A \geq \prod_{j=1}^{n} g_j > 0. \)

In the same way, we can show that the determinant of each of the principal submatrix of \( A \) is positive, therefore \( A \) is an \( M \)-matrix.

3. Proof of the Theorem. Ky Fan \([2]\) has shown that if a complex matrix \( A = (a_{ij}) \) and an \( M \)-matrix \( B = (b_{ij}) \) satisfy

\[ (3.1) \quad b_{ii} \leq |a_{ii}| \quad \text{for all } i \in N \]
and

\[ (3.2) \quad |a_{ij}| \leq |b_{ij}| \quad \text{for } i \neq j, \]

then \( |\det A| \geq \det B \). Our Theorem follows immediately, since the real matrix \( B = (b_{ij}) \), given by

\[ b_{ij} = |a_{ij}| \quad \text{if } i = j, \]
\[ = -|a_{ij}| \quad \text{if } i \neq j, \]

is an \( M \)-matrix (by Corollary 4), and in addition to (3.1) and (3.2), it satisfies

\[ (3.3) \quad \det B \geq \prod_{j=1}^{n} g_j > 0, \]
where the \( g_j \) are given in (2.1).

4. For the case where \( J = N \), Ostrowski ([3], [4]) has given the following lower bounds for \( |\det A| \):

\[ (4.1) \quad M_1 = \prod_{i=1}^{n} \left( |a_{ii}| - \sum_{j=i+1}^{n} |a_{ij}| \right), \]
and

\[ (4.2) \quad M_2 = \prod_{i=1}^{n} \left( |a_{ii}| - \sum_{j=1}^{i-1} |a_{ij}| \right). \]

Observe that, for \( J = N \), the \( g_j \) given in (2.1) are positive for any arbitrary permutation \((i_1 \ i_2 \cdots \ i_n)\) of \( N \). If we choose the permutation \((1 \ 2 \cdots \ n)\), then our lower bound \( \prod_{j=1}^{n} g_j = M_1 \). If the permutation is chosen to be \((n \ (n-1) \cdots \ 1)\), then \( \prod_{j=1}^{n} g_j = M_2 \). The following simple example shows that we have a better lower bound.
Example 1. Let

\[
A = \begin{pmatrix}
3 & 2 & 1 \\
2 & 5 & 2 \\
3 & 2 & 4
\end{pmatrix}
\]

Then \(M_1 = \frac{1}{2} \cdot 3 \cdot 4 = 6\), \(M_2 = 3 \cdot 3 \cdot \frac{1}{2} = \frac{9}{2}\) and choosing the permutation \((2 \ 3 \ 1)\), we get

\[
\prod_{i=1}^{3} g_i = 1 \cdot \frac{5}{2} \cdot 3 = \frac{15}{2}.
\]

In general, since for each possible permutation \((i_1 \ i_2 \ \cdots \ i_n)\) of \(N\) (obtained from Corollary 3) we can find a corresponding set of \(g_i\)'s, and thus we may have more than one lower bound for \(|\det A|\).

Example 2. Consider the diagonally dominant matrix

\[
A = \begin{pmatrix}
1 & \frac{1}{2} & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 \\
2 & 2 & 4 & 0 & 0 \\
0 & 3 & 3 & 9 & 3 \\
0 & 0 & 4 & 4 & 8
\end{pmatrix}
\]

Clearly \(A\) is reducible. Now, \(J = \{1\}\) and we have the following sequences of nonzero elements of \(A\):

\[
\{a_{21}\}, \quad \{a_{31}\}, \quad \{a_{42}, a_{21}\}, \quad \{a_{53}, a_{81}\}.
\]

Hence the matrix \(A\) satisfies the conditions of the Theorem.

References


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