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A SUFFICIENT CONDITION FOR NONVANISHING OF DETERMINANTS

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ABSTRACT. In this note we derive sufficient conditions for a diagonally dominant reducible matrix to be nonsingular.

1. Throughout this note we are concerned with $A=(a_{ij})$, an $n\times n$ matrix which is diagonally dominant and where

$$J = \left\{ i \in N \;\middle|\; |a_{ii}| > \sum_{j=1; j\neq i}^{n} |a_{ij}| \right\} \neq \varnothing \tag{1.1}$$

where $N=\{1,2,\cdots,n\}$. If $J=N$, A is strictly diagonally dominant and then the Gersgorin circle theorem implies that the determinant of A does not vanish [**1**, p. 106]. If A is irreducible, Taussky [**5**] has shown that A is nonsingular. In this note, we prove the following theorem.

THEOREM. *Let the matrix A be such that for each $i\notin J$ there is a sequence of nonzero elements of A of the form $a_{ii_1}, a_{i_1i_2}, \cdots, a_{i_rj}$ with $j\in J$. Then A is nonsingular.*

2. We need the following lemma and results to prove the Theorem.

LEMMA 1. *Let A satisfy the conditions of the theorem. Then for any nonempty subset L of N such that $L\cap J=\varnothing$, there is a nonzero element a_{ij} with $i\in L$ and $j\notin L$.*

PROOF. Let L be a nonempty subset of N such that $L\cap J=\varnothing$. Choose $i_1\in L$, then $i_1\notin J$ and, hence, there is a sequence of nonzero elements of A of the form $a_{i_1i_2}, a_{i_2i_3}, \cdots, a_{i_{s-1}i_s}$ for some $i_s\in J$. Let r be the first integer such that $i_r\notin L$ and note that $2\leq r\leq s$ since $i_1\in L$ and $i_s\notin L$. Then $a_{i_{r-1}i_r}\neq 0$ with $i_{r-1}\in L$ and $i_r\notin L$. This proves Lemma 1.

COROLLARY 2. *Let A satisfy the conditions of the Theorem. If $J=\{i_1, i_2, \cdots, i_k\}$, then there is a permutation $(i_1\, i_2 \cdots i_n)$ of N such that, for each $j=k+1,\cdots,n$, $a_{i_ji_l}\neq 0$ for some $l<j$.*

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PROOF. If $J=N$, then there is nothing to prove. Suppose $J\neq N$, then $L_1=N-J$ is nonempty and $L_1\cap J=\varnothing$. Hence, by Lemma 1, there are an $i_{k+1}\in L_1$ and $j\notin L_1$ such that $a_{i_{k+1}j}\neq 0$. Since $j\notin L_1$, we have $j\in N-L_1=J$. Hence $j=i_l$ for some $1\leqq l\leqq k$.

Let $L_2=L_1-\{i_{k+1}\}$. If $L_2=\varnothing$, then the proof is completed. Suppose $L_2\neq\varnothing$. Obviously $L_2\cap J=\varnothing$. Again, by Lemma 1, there are $i_{k+2}\in L_2$ and $j\notin L_2$ such that $a_{i_{k+2}j}\neq 0$. Since $j\notin L_2$, we have $j\in J\cup\{i_{k+1}\}$. Hence $j=i_l$ for some $1\leqq l\leqq k+1$. The corollary is proved by repeating the above process until $L_p=L_{p-1}-\{i_{k+p-1}\}=\varnothing$.

COROLLARY 3. *Let A satisfy the conditions of the Theorem. If $J=\{i_1, i_2, \cdots, i_k\}$, then there is a permutation $(i_1\, i_2 \cdots i_n)$ of N such that*

$$g_j = |a_{i_j i_j}| - \sum_{l=j+1}^{n} |a_{i_j i_l}|, \qquad j = 1, 2, \cdots, n, \tag{2.1}$$

are positive.

PROOF. From Corollary 2, there is a permutation $(i_1\, i_2 \cdots i_n)$ of N such that, for each $j=k+1, \cdots, n$, $a_{i_j i_l}\neq 0$ for some $l<j$. Notice that if $i_j\in J$, then

$$g_j = |a_{i_j i_j}| - \sum_{l=j+1}^{n} |a_{i_j i_l}| \geqq |a_{i_j i_j}| - \sum_{l=1; l\neq j}^{n} |a_{i_j i_l}| > 0.$$

Hence $g_j>0$, $j=1, 2, \cdots, k$. For $j=k+1, \cdots, n$,

$$\begin{aligned} g_j &= |a_{i_j i_j}| - \sum_{l=j+1}^{n} |a_{i_j i_l}| \\ &> |a_{i_j i_j}| - \sum_{l=j+1}^{n} |a_{i_j i_l}| - \sum_{l=1}^{j-1} |a_{i_j i_l}| \geqq 0 \end{aligned}$$

since $a_{i_j i_l}\neq 0$ for some $l<j$. This completes the proof.

COROLLARY 4. *Let A satisfy the conditions of the Theorem. If A is real and $a_{ij}\leqq 0$, $a_{jj}>0$, then A is an M-matrix* [**2**].

PROOF. Let $J=\{i_1, i_2, \cdots, i_k\}$. Then it follows from Corollary 3 that there is a permutation $(i_1\, i_2 \cdots i_n)$ of N such that

$$g_j = |a_{i_j i_j}| - \sum_{l=j+1}^{n} |a_{i_j i_l}| > 0, \qquad j = 1, 2, \cdots, n.$$

Now, $\det A=g_1 \det A(i_1)+\det A\{i_1\}$, where $A(i, j, \cdots, k)$ denotes the matrix obtained from A with i-, j-, $\cdots$, k-rows and columns deleted, and $A\{i_j\}$ is the matrix A with the entry $a_{i_j i_j}$ replaced by $\sum_{l=j+1}^{n} |a_{i_j i_l}|$. From [**1**, p. 294, Problem 9], we have $\det A\{i_1\}\geqq 0$. Thus

$$\det A \geqq g_1 \det A(i_1).$$

Similarly, we have

$$\det A(i_1) \geqq g_2 \det A(i_1, i_2),$$
$$\vdots$$
$$\det A(i_1, i_2, \cdots, i_{n-1}) \geqq g_n.$$

Hence $\det A \geqq \prod_{j=1}^{n} g_j > 0$.

In the same way, we can show that the determinant of each of the principal submatrix of A is positive, therefore A is an M-matrix.

3. **Proof of the Theorem.** Ky Fan [2] has shown that if a complex matrix $A=(a_{ij})$ and an M-matrix $B=(b_{ij})$ satisfy

$$b_{ii} \leqq |a_{ii}| \quad \text{for all } i \in N \tag{3.1}$$

and

$$|a_{ij}| \leqq |b_{ij}| \quad \text{for } i \neq j, \tag{3.2}$$

then $|\det A| \geqq \det B$. Our Theorem follows immediately, since the real matrix $B=(b_{ij})$, given by

$$\begin{aligned} b_{ij} &= |a_{ij}| \quad \text{if } i = j, \\ &= -|a_{ij}| \quad \text{if } i \neq j, \end{aligned}$$

is an M-matrix (by Çorollary 4), and in addition to (3.1) and (3.2), it satisfies

$$\det B \geqq \prod_{j=1}^{n} g_j > 0, \tag{3.3}$$

where the g_j are given in (2.1).

4. For the case where $J=N$, Ostrowski ([3], [4]) has given the following lower bounds for $|\det A|$:

$$M_1 = \prod_{i=1}^{n} \left(|a_{ii}| - \sum_{j=i+1}^{n} |a_{ij}| \right), \tag{4.1}$$

and

$$M_2 = \prod_{i=1}^{n} \left(|a_{ii}| - \sum_{j=1}^{i-1} |a_{ij}| \right). \tag{4.2}$$

Observe that, for $J=N$, the g_j given in (2.1) are positive for any arbitrary permutation $(i_1\, i_2 \cdots i_n)$ of N. If we choose the permutation $(1\; 2 \cdots n)$, then our lower bound $\prod_{i=1}^{n} g_i = M_1$. If the permutation is chosen to be $(n\,(n-1) \cdots 1)$, then $\prod_{i=1}^{n} g_i = M_2$. The following simple example shows that we have a better lower bound.

EXAMPLE 1. Let

$$A = \begin{pmatrix} 3 & 2 & \frac{1}{2} \\ 2 & 5 & 2 \\ \frac{3}{2} & 2 & 4 \end{pmatrix}$$

Then $M_1 = \frac{1}{2} \cdot 3 \cdot 4 = 6$, $M_2 = 3 \cdot 3 \cdot \frac{1}{2} = \frac{9}{2}$ and choosing the permutation (2 3 1), we get

$$\prod_{i=1}^{3} g_i = 1 \cdot \frac{5}{2} \cdot 3 = \frac{15}{2}.$$

In general, since for each possible permutation $(i_1\, i_2 \cdots i_n)$ of N (obtained from Corollary 3) we can find a corresponding set of g_j's, and thus we may have more than one lower bound for $|\det A|$.

EXAMPLE 2. Consider the diagonally dominant matrix

$$A = \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 0 \\ 0 & 3 & 3 & 9 & 3 \\ 0 & 0 & 4 & 4 & 8 \end{pmatrix}$$

Clearly A is reducible. Now, $J = \{1\}$ and we have the following sequences of nonzero elements of A:

$$\{a_{21}\}, \quad \{a_{31}\}, \quad \{a_{42}, a_{21}\}, \quad \{a_{53}, a_{31}\}.$$

Hence the matrix A satisfies the conditions of the Theorem.

REFERENCES

1. R. Bellman, *Introduction to matrix analysis*, McGraw-Hill, New York, 1960. MR **23** #A153.

2. Ky Fan, *Inequalities for sum of two M-matrices*, Inequalities I, Academic Press, New York and London, 1967, pp. 105–117. Edited by Oved Shisha.

3. A. Ostrowski, *On some conditions for nonvanishing of determinants*, Proc. Amer. Math. Soc. **12** (1961), 268–273. MR **25** #1168.

4. ———, *Note on bounds for determinants with dominant principal diagonal*, Proc. Amer. Math. Soc. **3** (1952), 26–30. MR **14**, 611.

5. O. Taussky, *A recurring theorem on determinants*, Amer. Math. Monthly **56** (1949), 672–676. MR **11**, 307.

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