AN EXTREMAL PROBLEM FOR POLYNOMIALS WITH A PRESCRIBED ZERO

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Abstract. Let \( P_{n,b} \) denote the class of all polynomials \( p_n(z) \) of degree at most \( n \) in \( z \) which satisfy \( \max_{|z|=1} |p_n(z)| = 1 \), and \( |p_n(1)| = b \), \( 0 \leq b < 1 \). Let \( c \in (0, n] \), and set

\[
\mu_b(c, n) = \sup_{p_n \in P_{n,b}} \left\{ \min_{|z|=1-\varepsilon/n} |p_n(z)| \right\}
\]

Upper estimates for \( \mu_b(c, n) \) are obtained.

Let \( U \) denote the open unit disc in the complex \( z \) plane, \( T \) its boundary, and let \( P_{n,0} \) denote the class of all polynomials \( p_n(z) \) of degree at most \( n \) in \( z \), satisfying \( \max_{z \in T} |p_n(z)| = 1 \) and \( p_n(1) = 0 \). The extremal problem in question is to estimate

\[
\mu(c, n) = \sup_{p_n \in P_{n,0}} \left\{ \min_{|z|=1-c/n} |p_n(z)| \right\},
\]

where \( 0 < c \leq n \). This problem was mentioned by Professor Paul Erdős during a lecture at the University of Montreal in July, 1971. He attributed the problem to G. Halász, of the Mathematical Institute of the Hungarian Academy of Sciences; Erdős asked if there exists a constant \( c \) such that

\[
\mu(c, n) = 1 - \varepsilon_n \text{ where } \varepsilon_n \to 0 \text{ as } n \to \infty.
\]

It is easily seen that no such constant \( c \) exists. In fact, if \( p_n \in P_{n,0} \), then also \( q_n(z) = z^n p_n(1/z) \), and by S. Bernstein’s theorem [3, p. 45] on the derivative of a polynomial, \( |q'_n(z)| \leq n \) for \( z \in T \). Hence it follows that \( |z^{n-1} q'_n(1/z)| \leq n \) for \( z \in T \) and by the maximum principle, also for all \( z \in U \). Replacing \( z \) by \( 1/z \) we find that

\[
|q'_n(z)| \leq n |z|^{n-1} \text{ for all } |z| \geq 1.
\]

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Consequently,

\[ |q_n((1 - c/n)^{-1})| = \left| \int_1^{(1-c/n)^{-1}} q'_n(t) \, dt \right| \leq (1 - c/n)^{-n} - 1 = (1 - c/n)^{-n}(1 - (1 - c/n)^n), \]

that is,

\[ |p_n(1 - c/n)| = |(1 - c/n)^n q_n((1 - c/n)^{-1})| \leq 1 - (1 - c/n)^n \to 1 - e^{-c} \text{ as } n \to \infty. \]

The inequality (1) provides a negative answer to the question raised by Erdős and also gives an upper estimate for \( \mu(c, n) \). However, this estimate is quite crude. The following theorem, which we shall prove, gives "essentially" best possible upper estimates for \( \mu(c, n) \).

**Theorem 1.** In the above notation,

\[ \mu(c, n) < \frac{1 - (1 - c/n)^n}{1 + (1 - c/n)^n} \text{ if } 0 < c \leq 1, \]

and

\[ \mu(c, n) < \frac{(2n - 1)c - (2n - c)(1 - 1/n)^n}{(2n - 1)c + (2n - c)(1 - 1/n)^n} \text{ if } 1 < c \leq n. \]

The right-hand side of (2) is equal to \( c/2 + o(c) \) as \( c \to 0 \); moreover, the polynomial \( p_n(z) = (1 - z^n)/2 \) satisfies

\[ \min_{|z|=1-c/n} |p_n(z)| = |p_n(1 - c/n)| = \frac{1 - (1 - c/n)^n}{2} = c/2 + o(c) \]

as \( c \to 0 \). Consequently, the inequality (2) is the best possible in the limit as \( c \to 0 \).

We find from (3) that \( \mu(c, n) \leq 1 - 1/ec + o(1/c) \) as \( c \to \infty \). We shall show that the function \( 1/(ec) \) cannot be replaced by one which approaches zero more slowly with regards to order, as \( c \to \infty \). We prove

**Theorem 2.** Given

\[ \lambda > \frac{1}{\pi} \int_{-\infty}^{\infty} \left| \log \left( 1 - \frac{\sin^2 u}{u^2} \right) \right| \, du \]

there exists a positive number \( A(\lambda) \), depending only on \( \lambda \), such that whenever \( c > A(\lambda) \), then

\[ \mu(c, n) > \exp(-\lambda/c) > 1 - \lambda/c. \]

For the proof of Theorem 1 we use two subsidiary results.

**Lemma 1 [1, Theorem 4].** Let \( D \) be a circular domain in the \( z \)-plane, and \( S \) an arbitrary set of points in the \( w \)-plane. If the polynomial \( p_n \) of degree
n satisfies $p_n(z) = w \in S$ for all $z \in D$, then for all $z \in D$ and all $\zeta \in D$,
\[
\frac{\zeta p_n'(z)}{n} + p_n(z) - \frac{z p_n'(z)}{n} \in S.
\]

**Lemma 2.** If $f(z)$ is analytic in $U$, where it satisfies $|f(z)| \leq 1$, then for $0 \leq \alpha < 2\pi$ and $0 \leq r_1 < r_2 < 1$,
\[
(4) \quad f(r_1 e^{i\alpha}) \leq (A - B)/(A + B)
\]
where
\[
A = (1 + r_2)(1 - r_1)\{1 + |f(r_2 e^{i\alpha})|\},
\]
\[
B = (1 - r_2)(1 + r_1)\{1 - |f(r_2 e^{i\alpha})|\}.
\]

**Proof of Lemma 2.** It is well known that if $f(z)$ is analytic in $U$, where it satisfies $|f(z)| \leq 1$, then
\[
|f'(z)|/(1 - |f(z)|^2) \leq 1/(1 - |z|^2) \quad \text{for all } z \in U.
\]
Hence
\[
\left| \int_{r_1}^{r_2} \frac{|f(re^{i\alpha})|}{1 - |f(re^{i\alpha})|^2} \, dr \right| \leq \int_{r_1}^{r_2} \frac{|f'(re^{i\alpha})|}{1 - |f(re^{i\alpha})|^2} \, dr \leq \int_{r_1}^{r_2} \frac{dr}{1 - r^2}.
\]
Now if $|f(r_1 e^{i\alpha})| > |f(r_2 e^{i\alpha})|$, we get
\[
\frac{1 + G_1}{1 - G_1} / \frac{1 + G_2}{1 - G_2} \leq \frac{1 + r_2}{1 - r_2} / \frac{1 + r_1}{1 - r_1}
\]
where $G_k = |f(r_k e^{i\alpha})|$, $k = 1, 2$, which readily gives the desired estimate of $|f(r_1 e^{i\alpha})|$. The inequality (4) is trivially true if $|f(r_1 e^{i\alpha})| \leq |f(r_2 e^{i\alpha})|$. 

**Proof of Theorem 1.** Let $p_n \in \mathscr{Q}_{n,0}$, $0 < c \leq 1$, and let
\[
\min_{|z| = 1 - c/n} |p_n(z)| = a.
\]
We wish to show that
\[
a < \{1 - (1 - c/n)^n\}/\{1 + (1 - c/n)^n\}.
\]
Without loss of generality we may suppose that $p_n(z) \neq 0$ in $U$, and therefore
\[
\min_{|z| \leq 1 - c/n} |p_n(z)| = \min_{|z| = 1 - c/n} |p_n(z)| = a.
\]
This implies that $p_n$ maps the circular domain $D = \{z: |z| \leq 1 - c/n\}$ onto a set $S$ which lies in the ring $\{w: a \leq |w| < 1\}$. Hence by Lemma 1,
\[
(1 - c/n)|p_n'(z)|/n < (1 - a)/2.
\]
for all $|z| = 1 - c/n$, i.e.,

$$|p_n((1 - c/n)z)| < \frac{1}{2}(1 - a)n^2/(n - c) \quad \text{for all } |z| = 1.$$ 

The same inequality holds for the polynomial $z^{n-1}p_n((1-c/n)/z)$. Using the maximum modulus principle, we therefore conclude that

$$|z^{n-1}p_n((1-c/n)/z)| \leq \frac{1}{2}(1 - a)n^2/(n - c) \quad \text{for all } z \in (U \cup T).$$

Replacing $z$ by $(1 - c/n)/z$ we obtain

$$|p_n(z)| < \frac{1}{2}(1 - a)n^2/(n - c) \{z/(1 - c/n)\}^{n-1} \quad \text{for all } |z| \geq 1 - c/n.$$ 

This implies that

$$0 = \lim_{t \to 0} \left| p_n(t) - p_n(1) \right| = \left| p_n(1 - c/n) + \int_{1-c/n}^{1} p_n'(t) \, dt \right| \geq a - \int_{1-c/n}^{1} \frac{1}{2}(1 - a)n^2/(n - c) \{t/(1 - c/n)\}^{n-1} dt \geq a - \frac{1}{2}(1 - a)((1 - c/n)^{-n} - 1),$$

or $a < (1 - (1 - c/n)^n)/(1 + (1 - c/n)^n)$. This establishes the relation (2).

The above proof is valid for $0 < c < n$; however, for $c > 1$, the estimate just obtained is not as good as the estimate (3). In order to prove (3) we apply (4) with $f(z) = p_n(z)$, $r_1 = 1 - c/n$ where $1 < c \leq n$, $r_2 = 1 - 1/n$ and $z = r^*$ where $|p_n(z)|$ attains its minimum on the circle $\{z:|z|=1-1/n\}$ at the point $z=(1-1/n)e^{i\alpha^*}$. We get

$$\min_{|z|=1-c/n} |p_n(z)| \leq |p_n((1 - c/n)e^{i\alpha^*})| < \frac{(2n - 1)c - (2n - c)(1 - 1/n)^n}{(2n - 1)c + (2n - c)(1 - 1/n)^n}$$

which completes the proof of Theorem 1.

**Proof of Theorem 2.** We consider the nonnegative trigonometric polynomial

$$t(\theta) = (n + 1)^{-2}[n(n + 1) - 2\{n \cos \theta + (n - 1)\cos 2\theta + \cdots 
+ 2 \cos(n - 1)\theta + \cos n\theta\}]$$

$$\equiv 1 - \frac{1}{(n + 1)^2} \left( \sin(n + 1)\theta/2 \right)^2$$

of degree $n$ vanishing at $\theta = 0$. There exist (see [2, p. 117]) polynomials $p_n \in \mathcal{P}_{n,\theta}$ such that

$$|p_n(e^{i\theta})|^2 = t(\theta).$$

Amongst the various polynomials $p_n$ satisfying (5) there is one (except for a constant factor of unit modulus) which does not vanish in $U$. If we
denote it by $p_n^*(r\exp(\theta))$, then for $r < 1$ and $-\pi \leq \varphi < \pi$

$$|p_n^*(r\exp(\theta))| = \exp\left(\frac{1}{4\pi} \int_{-\pi}^{\pi} \log |p_n^*(e^{i\theta})|^2 \frac{1 - r^2}{1 - 2r \cos(\theta - \varphi) + r^2} \, d\theta\right).$$

Thus

$$|p_n^*((1 - c/n)\exp(\theta))| = \exp(I_n(\varphi)),$$

where

$$I_n(\varphi) = \frac{1}{4\pi} (cn - \frac{1}{4}c^2) \int_{-\pi/2}^{\pi/2} \log |p_n^*(e^{i\theta})|^2 \frac{d\theta}{\frac{1}{4}c^2 + (n^2 - cn)\sin^2(\theta - \frac{1}{2}\varphi)}.$$

It can be shown that for $0 \leq \theta \leq \pi/2$,

$$|p_n^*(e^{i\theta})|^2 = 1 - \frac{1}{(n + 1)^2} \left(\frac{\sin((n + 1)\theta)^2}{\sin \theta}\right) = (1 + \gamma_n)D((n + 1)\theta)$$

where $D(u) = 1 - (\sin^2 u)/u^2$ and $|\gamma_n| < 5/(n+1)^2$. Hence

$$I_n(\varphi) = -\frac{1}{4\pi} (cn - \frac{1}{4}c^2) \int_{-\pi/2}^{\pi/2} |\log D((n + 1)\theta)| \frac{d\theta}{\frac{1}{4}c^2 + (n^2 - cn)\sin^2(\theta - \frac{1}{2}\varphi)} + \delta_n,$$

where $|\delta_n| < 10/cn$ if $n \geq 3$. Since the right-hand side of (6) is decreased when $c^2/4 + (n^2 - cn)\sin^2(\theta - \varphi/2)$ is replaced by $c^2/4$ we conclude that

$$I_n(\varphi) > -\frac{1}{\pi c} (n - \frac{1}{4}c) \int_{-\pi/2}^{\pi/2} |\log D((n + 1)\theta)| \, d\theta - |\delta_n|$$

$$> -\frac{1}{\pi c} \frac{n - \frac{1}{4}c}{n + 1} \int_{-\infty}^{\infty} |\log D(u)| \, du - |\delta_n|,$$

from which the statement of Theorem 2 follows.

With reference to the problem of Halász, it is natural to define a more general class $\mathcal{P}_{n,b}$ of polynomials $p_n(z)$ which are of degree at most $n$ in $z$, satisfying $\max_{z \in T} |p_n(z)| = 1$, and $|p_n(1)| = b$ where $b \in [0, 1)$, and to estimate

$$\mu_b(c, n) = \sup_{p_n \in \mathcal{P}_{n,b}} \left\{ \min_{|z| = 1 - c/n} |p_n(z)| \right\}.$$

Our proof of Theorem 1 applies with slight modification, to give the following result.

**THEOREM 1'.** If $p_n \in \mathcal{P}_{n,b}$, then for $0 < c < n$,

$$\min_{|z| = 1 - c/n} |p_n(z)| < \frac{1 - (1 - 2b)(1 - c/n)^n}{1 + (1 - c/n)^n}.$$
Furthermore, if $c_0 \in (0, n)$ is arbitrary, and if $c_0 \leq c \leq n$, then

$$\min_{|z|=1-c/n} p(z) < \frac{A + \{2nb(c + c_0 - cc_0/n) - B\}(1 - c_0/n)^n}{A + \{2nb(c - c_0) + B\}(1 - c_0/n)^n}$$

where $A = (2n-c_0)c$, $B = (2n-c)c_0$.

In analogy with the problem of Halász, or the more general case just considered, let $\mathcal{F}_{n,b}$ denote the class of all polynomials $p_n(z)$ of degree at most $n$ in $z$ which satisfy $\max_{z \in T} \text{Re} p_n(z) = 1$, and $\text{Re} p_n(1) = b$, where $b \in [0, 1)$.

**Theorem 1".** If $p_n \in \mathcal{F}_{n,b}$, then

$$\min_{|z|=1-c/n} \text{Re} p_n(z) < B(c) = \frac{1 - (1 - 2b)(1 - c/n)^n}{1 + (1 - c/n)^n}.$$  

Furthermore, for any fixed $c_1 \in (0, n)$ and for $c_1 \leq c \leq n$,

$$\min_{|z|=1-c/n} \text{Re} p_n(z) < 1 + \log\left(\frac{1 - (2n - c)c_1 e - e^A}{(2n - c_1)c e + e^A}\right)$$

where $A = B(c_1)$.

**Sketch of Proof.** The inequality (7) can be proved in the same way as (2). If $\text{Re} p_n(z)$ attains its minimum on the circle $\{z: |z| = 1 - c_1/n\}$ at $z = (1 - c_1/n)e^{i\alpha_1}$, then for $c_1 < c \leq n$, we may apply Lemma 2 with $f(z) = \text{exp}\{p_n(z)\} - 1$, $r_1 = 1 - c/n$, $r_2 = 1 - c_1/n$ and $\alpha = \alpha_1$, to get

$$\text{exp} \text{Re}\{p_n((1 - c/n)e^{i\alpha_1}) - 1\} \leq (B - C)/(B + C)$$

where

$$B = (2n - c_1)c[1 + \text{exp}\{\text{Re}((1 - c_1/n)e^{i\alpha_1}) - 1\}],$$

$$C = (2n - c)c_1[1 - \text{exp}\{p_n((1 - c_1/n)e^{i\alpha_1}) - 1\}].$$

The inequality (8) now follows from this, in view of the definition of $A$, and since

$$\min_{|z|=1-c/n} \text{Re} p_n(z) \leq \text{Re} p_n((1 - c/n)e^{i\alpha_1}).$$

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**References**


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