INVERTIBLE MEASURE PRESERVING TRANSFORMATIONS
AND POINTWISE CONVERGENCE

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Abstract. An investigation of pointwise convergence of sequences \( \{ \sum_{j=-\infty}^{\infty} a_j f(T^j x) : k=1, 2, \cdots \} \) where \( f \) lies in the space \( L^1([0, 1]) \) of Lebesgue integrable functions on the unit interval, \( T \) is an invertible measure preserving transformation on \([0, 1]\), and the sequence of polynomials \( \{ \sum_{j=-\infty}^{\infty} a_j z^{-j} : k=1, 2, \cdots \} \) is uniformly bounded and pointwise convergent for all \( z \) such that \( |z|=1 \).

Spectral properties. An invertible measure preserving transformation \( T \) on the unit interval \( I \) is known to induce a unitary operator on the space \( L^2(I) \) of square integrable functions on \( I \) [6, p. 13]. By the spectral theorem [5, p. 71] there exists a spectral measure \( E \) on the Borel subsets of the unit circle \( C \) in the complex plane such that for any integer \( k \), \( U^k = \int z^k E(dz) \) in the sense of strong convergence. Let the resolution of the identity \( E_t, t \in [0, 2\pi] \), be given by \( E(\{ \exp(is) : 0 \leq s < t \}) \). Then [3, p. 482]

\[
E_t = \sum_{j \neq 0} \frac{\exp(ijt) - 1}{2\pi ij} U^{-j} + \frac{t}{2\pi} + \frac{E(\{1\}) - E(\{\exp(it)\})}{2}
\]

where, for each \( z \) in \( C \), \( E(\{z\}) = \lim(\sum_{j=-n}^{n} z^j U^{-j})/(2n+1) \) and the symbol \( \sum_{j \neq 0} \) denotes the limit as \( n \) tends to infinity of the sum \( \sum_{j=-n}^{n} z^j U^{-j} \).

Substituting the Fourier series

\[
\pi - \sum_{j \neq 0} \frac{\exp(ijs)}{ij} = s, \quad 0 < s < 2\pi,
\]

\[
\pi = s, \quad s = 0,
\]

on the right-hand side of the identity [1, p. 100]

\[
s = \pi + 32 \sum_{j=0}^{\infty} \frac{\sin(\frac{1}{2}(2j + 1)s) - (-1)^j \cos(\frac{1}{2}(2j + 1)s)}{\pi^2(2j + 1)^3} \quad (0 \leq s \leq 2\pi),
\]

and then integrating both sides with respect to the spectral measure.
for the unitary operator \( \exp(-it)U \) yields

\[
E_t = \frac{t}{2\pi} + \frac{1}{2} E(\{1\}) - \sum_{j \neq 0} \frac{U^{-j} \sum_{j=0}^{\infty} \exp(-i(2j+1)t) U^{(2j+1)/4}}{(2j+1)^3} - \frac{16}{\pi^3} \sum_{j=0}^{\infty} (-1)^j \left\{ \frac{\exp(-i(2j+1)t) U^{(2j+1)/4}}{(2j+1)^3} \right\}
\]

By the uniform boundedness of the series [7, p. 18] we can justify taking the integral inside the summation signs above.

The unitary operators \( U^{k/4}, k=0, \pm 1, \pm 2, \cdots \), are defined by \( U^{k/4} = \int z^{k/4} E(dz) \). Thus the convolution property for the spectral measure of a unitary operator with the multiplicative property [4, pp. 639, 640] permits us to establish that, since \( U \) is multiplicative, then so is \( U^{k/4} \).

For if \( f, g \) and their product \( fg \) lie in \( L^2(I) \) then

\[
U^{k/4}fg = \int z^{k/4} E(dz)fg = \int \int z^{k/4} E(w^{-1} dw) fE(dw)g
\]

Hence if \( f \) lies in \( L^2(I) \) with \( L^1 \) norm \( \|f\|_1 \) then there exists \( g \) in \( L^2(I) \) with \( L^2 \) norm \( \|g\|_2 \) such that \( f=g^2, \|f\|_1=\|g\|_2^2, \) and \( \|U^{k/4}f\|_1=\|U^{k/4}g\|_2^2=\|f\|_1 \). Using the identity above for \( E_t \) it now follows that there exists a constant \( K \) such that for any collection \( \{B_m: m=1, 2, \cdots\} \) of disjoint half-open interval subsets of \( C \) and any \( f \) in \( L^2(I) \) we have \( \|E(\bigcup B_m)f\|_1 \leq K\|f\|_1 \). By the usual measure theoretic argument (Dinculeanu [2]), for any \( f \) in \( L^2(I) \) and any Borel subset \( B \) of \( C \), \( \|E(B)f\|_1 \leq K\|f\|_1 \). Since \( L^2(I) \) is dense in \( L^1(I) \) we extend by continuity the operator \( E \) to \( L^1(I) \) and so (retaining the symbol \( E \) for the extension) \( \|E(B)f\|_1 \leq K\|f\|_1 \) for all \( f \) in \( L^1(I) \) and Borel subset \( B \) of \( C \). Note that the space \( L^\infty(I) \) of essentially bounded functions on \( I \) lies in \( L^2(I) \). Hence \( E \) is defined on \( L^\infty(I) \). We now deduce that for any \( h \) in \( L^\infty(I) \) with \( L^\infty \) norm \( \|h\|_\infty \) and any Borel \( B \) in \( C \), \( \|E(B)h\|_\infty \leq K\|h\|_\infty \). For if \( f \) lies in \( L^1(I) \), using \( (f, h) \) to denote the integral of the product \( \bar{f}h \) (where \( \bar{h} \) is the complex conjugate of \( h \)) over \( I \), we get \( (E(B)f, h) = (f, E(B)h) \) which is clear if \( f \) lies in \( L^2(I) \) and extends to \( L^1(I) \) by continuity.

Next let us show the existence of a constant \( K' \) such that for any \( h \) in \( L^\infty(I) \) and any sequence \( \{B_k\} \) of disjoint Borel subsets of \( C \), \( \|\sum |E(B_k)h|\|_\infty \leq K'\|h\|_\infty \). Otherwise there would exist some finite family
\{B_k: k=1, 2, \cdots, n\} \) of disjoint Borel subsets of \( C \) such that for some \( h \) in \( L^2(I) \), \( \sum |E(B_k)h| \) is "much" greater than \( \|h\|_\infty \) on some subset \( X \) of \( I \) of positive measure. Hence by considering the real and imaginary parts of \( E(B_k)h \) and all possible subsequences of \( \{B_k: k=1, 2, \cdots, n\} \), we see that there must exist some subsequence \( \{B_{k_j}\} \) for which either the real or imaginary part of \( E(\bigcup B_{k_j})h \) is "much" greater in absolute value than \( \|h\|_\infty \) on a subset of \( X \) of positive measure, i.e. \( \|E(\bigcup B_{k_j})h\|_\infty > K\|h\|_\infty \) which is a contradiction.

By now we have that for any given \( h \) in \( L^\infty(I) \), \( E(\cdot)h(x) \) is a complex measure on the Borel subsets of \( C \) with total variation not exceeding \( K\|h\|_\infty \) [8, p. 117] for almost all \( x \) in \( I \). Hence we can define in the usual way the integral \( \int q(x, z)E(dz)h(x) \) of a bounded Borel measurable function \( q(x, z) \) on \( I \times C \) to yield an essentially bounded function of \( x \), i.e. an element of \( L^\infty(I) \). Furthermore if \( \{q_k(x, z)\} \) is a pointwise convergent uniformly bounded sequence of Borel measurable functions then by Lebesgue's dominated convergence theorem the integrals \( \int q_k(x, z)E(dz)h(x) \) form a uniformly bounded (in \( L^\infty(I) \)) almost everywhere pointwise convergent sequence of functions on \( I \).

**Convergence properties.** Consider a sequence of polynomials \( p_k(z) = \sum_{j=-\infty}^{\infty} a_j^k z^{-j}, k=1, 2, \cdots \), where \( z \) lies in \( C \) and \( a_j^k \) are complex coefficients all but a finite number of which vanish. For a given function \( f \) in \( L^1(I) \) define \( p_k(U)f \) to be \( \sum_{j=-\infty}^{\infty} U^{-j}(a_j^k f) \), i.e. \( \sum_{j=-\infty}^{\infty} a_j^k U^{-j}f \).

**Theorem.** If \( U \) is an operator on \( L^1(I) \) induced by an invertible measure preserving transformation on the unit interval \( I \) and \( \{p_k(z): k=1, 2, \cdots\} \) a pointwise convergent sequence of uniformly bounded (trigonometric) polynomials on the unit circle then, for all \( f \) in \( L^1(I) \), \( p_k(U)f(x) \) converges pointwise for almost all \( x \) in \( I \) as \( k \) tends to infinity.

**Proof.** If \( p_k(U)f \) does not converge pointwise almost everywhere, there exists a nonzero constant \( d \) such that for all \( x \) in a subset \( Y \) of \( I \) of positive measure \( |Y| \)

\[
\sup_{k', k'' \geq m} |p_{k'}(U)f(x) - p_{k''}(U)f(x)| > d
\]

for all integers \( m \). Hence given any \( m \) there exists an integer \( M \geq m \) and measurable functions \( k'(x), k''(x); m \leq k'(x), k''(x) \leq M \) such that for some function \( h, |h| = 1 \), we have

\[
\left( \sum_j U^{-j}(a_j^{k'(x)} - a_j^{k''(x)})f(x), h(x) \right) > \frac{d |Y|}{2}.
\]
Note that \( M \) was chosen to make
\[
\sup_{m \leq k', k' \leq M} |p_k(U)f(x) - p_{k'}(U)f(x)| > d
\]
for all \( x \) in a subset of \( Y \) of measure greater than \( |Y|/2 \). But by the measure preserving property of the operator inducing \( U \) we have
\[
\left( \sum_j U^{-j}(a_j^{k'(x)} - a_j^{k''(x)})f(x), h(x) \right) = \left( f(x), \sum_j (a_j^{k'(x)} - a_j^{k''(x)})U^j h(x) \right)
\]
and by the discussion at the end of the previous section this tends to zero as \( m \) tends to infinity, which is a contradiction. \( \text{Q.E.D.} \)

The above could be generalized to not necessarily invertible transformations on the real line, which would make Birkhoff's ergodic theorem [6, p. 18] a special case of the theorem above by taking the polynomial \( \sum_{j=1}^k z^{-j}(2k+1) \) for \( p_k(z) \). In fact we could go even further by considering operators which are \( L^1 \) and \( L^2 \) contractions with the multiplicative property by using the generalized spectral measures associated with them [9, pp. 12–18].

**BIBLIOGRAPHY**