SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

ON A CLASS OF ANALYTIC FUNCTIONS

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Abstract. We show that the class $\mathcal{E}_0$ of analytic functions $f$ in a plane region $\Omega \notin O_{AB}$ vanishing at $z_0 \in \Omega$ and such that $1/f$ omits a set of values of area $\geq \pi$ is not compact. Here $O_{AB}$ denotes the class of Riemann surfaces which have no nonconstant bounded analytic functions. We remark that the extremal functions maximizing $|f'(z_0)|$ in $\mathcal{E}_0$ coincide with linear transformations $w/(1-cw)$ of those for the same problem in the class $\mathcal{B}_0$ consisting of functions $f$ such that $f(z_0)=0$ and $|f(z)| \leq 1$, i.e. so-called Ahlfors functions. Here $1/c$ is an omitted value of the Ahlfors function.

Under the notations in the above abstract Ahlfors and Beurling [1] stated that the classes $\mathcal{B}_0$ and $\mathcal{E}_0$ are both compact and proved that the maxima of $|f'(z_0)|$ in $\mathcal{B}_0$ and $\mathcal{E}_0$ are equal. However, we can show that the alleged compactness of $\mathcal{E}_0$ is not true by constructing a counterexample: For the annulus $\frac{1}{2} < |z| < 2$, the functions $f_n = (\frac{1}{2})^n (z^n - (\frac{3}{2})^n)/z^n$, $n = 1, 2, \ldots$, belong to $\mathcal{E}_0$ with $z_0 = \frac{3}{2}$. Then $\{f_n\}$ tends to zero for $\frac{3}{2} + \delta < |z| < 2$ and to infinity for $\frac{3}{2} < |z| < \frac{5}{2} - \delta$, $\delta > 0$ as $n \to \infty$.

If $\Omega \notin O_{AB}$, there exist the extremal functions $A(z)$ which maximize $|f'(z_0)|$ in $\mathcal{B}_0$. Those functions are called the Ahlfors functions which are unique save for rotations [3]. If $1/c$ is an omitted value of $A(z)$, $A(z)/(1-cA(z))$ belongs to $\mathcal{E}_0$. By the result cited above it is extremal for the problem in $\mathcal{E}_0$.

For any extremal $g \in \mathcal{E}_0$, let $E$ be the set of all omitted values of $g$. From the extremality of $g$ the area of $E$ is equal to $\pi$. They used a transformation

$$\Phi \left( \frac{1}{g} \right) = \frac{1}{\pi} \int \int \frac{du \, dv}{\frac{1}{g} - w}, \quad w = u + iv,$$

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and proved $\Phi(1/g) \in \mathcal{B}_0$ with $\Phi'(1/g(z_0)) = g'(z_0)$ [1]. Hence $\Phi(1/g)$ is an Ahlfors function and there exists a point $\omega \in E$ such that $|\Phi(\omega)| = 1$. Note that $\Phi(w)$ is continuous in the whole plane [2]. Without loss of generality (by a rotation) we set $\Phi(\omega) = 1$ and $E_+ = E \cap \{\Re(w - \omega) \geq 0\}$. Then from the equality statement for Schwarz’s inequality in their proof we infer that $E_+$ coincides with the disc $r \leq 2 \cos \theta, |\theta| \leq \pi/2$ ($w - \omega = re^{i\theta}$), except for a set of area zero. We can deduce, from $\Phi(\omega) = 1$, that the area of $E - E_+$ vanishes. Denoting by $c$ the center of the above disc, by a direct calculation we see that $\Phi(w)$ reduces to a linear transformation $1/(w - c)$ for $|w - c| \geq 1$. Hence we have $g = \Phi(1/g)/(1 + c\Phi(1/g))$. Clearly $-1/c$ is an omitted value of the Ahlfors function $\Phi(1/g)$ and therefore $g$ is of the form stated in the abstract.

REFERENCES


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