THE CODIMENSION OF THE BOUNDARY OF A LATTICE IDEAL

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Abstract. In a compact connected topological lattice of finite codimension \( n \), the boundary of a proper principal ideal has codimension less than \( n \). It follows that the boundaries of proper intervals also have codimension less than \( n \).

In a topological lattice \( L \) the boundary of a principal (dual) ideal is a join (meet) subsemilattice of \( L \) [1, Lemma 5]. The purpose of this paper is to show that this semilattice has codimension less than \( n \) whenever \( L \) is a compact, connected topological lattice of codimension \( n \). Note that such a lattice has breadth \( n \) [5, Corollary 2.4].

If \( S \) is a semilattice and \( x \in S \), then \( M(x) = \{y \in S : x \leq y\} \); the set \( L(x) \) is defined dually; if \( x \leq y \), then \( [x, y] = M(x) \cap L(y) \). The interior and the closure of \( A \) are denoted by \( A^o \) and \( A^* \) respectively. The boundary of \( A \) is \( F(A) = A^* \setminus A^o \). If \( a \in S \) a topological semilattice and \( x \in F(M(a)) \), then \( [a, x] \subseteq F(M(a)) \).

Lemma. Let \( L \) be a compact, connected topological lattice of finite codimension. If \( a \in L \), then \( F(M(a)) \) is locally connected.

Proof. Let \( x \in F(M(a)) \) and let \( U \) be an open subset of \( F(M(a)) \) which contains \( x \). It is known that \( L \) has a basis of open convex sublattices [6, Theorem 5]. Hence we may choose an open convex sublattice \( V \) containing \( x \) such that \( V \cap F(M(a)) \subseteq U \). Let \( p, q \in V \cap F(M(a)) \); then the connected set \( [p \land q, p] \cup [p \land q, q] \) is contained in \( V \cap F(M(a)) \). Thus \( V \cap F(M(a)) \) is a connected open subset of \( F(M(a)) \) which contains \( x \) and which is contained in \( U \). Therefore \( F(M(a)) \) is locally connected.

We shall state and prove the main result in terms of principal dual ideals.

Theorem. Let \( L \) be a compact, connected topological lattice of finite codimension \( n \) and let \( a \in L \setminus \{0, 1\} \). Then the codimension of \( F(M(a)) \) is less than \( n \).
Proof. Since the result clearly holds for $n \leq 1$, we shall assume $n > 1$.

We shall first show that $x \in F(M(a))$ implies $[a, x]$ has codimension less than $n$. Let $x \in F(M(a))$ and let $U(a) = L \setminus [L(a) \cup M(a)]$. If $U(a) = \emptyset$, then $a$ is a cut point of $L$ [2, Theorem 1]. Thus $a = x$ and $[a, x] = \{a\}$ has codimension 0.

Suppose $U(a) \neq \emptyset$ and $x \neq a$. We shall construct an increasing net in $L(x) \cap U(a)$ which converges to $x$. Since $x \notin L(a)$ we may choose an open set $V$ containing $x$ such that $V \cap L(a) = \emptyset$. Now $x \in F(M(a))$ implies $V \cap L \setminus M(a) \neq \emptyset$. Note that $L \setminus M(a) = [L(a) \cup U(a)] \setminus \{a\}$ so that $V \cap U(a) \neq \emptyset$. Thus we may choose a net $\{x'_\alpha\}_{\alpha \in \Delta} \subseteq U(a)$ which converges to $x$. Let $y'_\alpha = x \wedge x'_\alpha$ for all $\alpha \in \Delta$. Then $\{y'_\alpha\}_{\alpha \in \Delta} \subseteq L(x)$ and converges to $x$. If $\{y'_\alpha\}_{\alpha \in \Gamma} \subseteq L(a)$ for any cofinal subset $\Gamma \subseteq \Delta$, then $\{(y'_\alpha, a)\}_{\alpha \in \Gamma}$ converges to $(x, a) \in L \times L$. Since the graph of $\leq$ is a closed subset of $L \times L$, then $x \leq a$. But $x \in M(a)$ so that $x = a$ contrary to the choice of $x \neq a$. Hence we may assume $y'_\alpha \notin L(a)$ for all $\alpha \in \Delta$. If $y'_\alpha \in M(a)$, then $a \leq y'_\alpha = x \wedge x'_\alpha \leq x'_\alpha$ contrary to $a \leq x'_\alpha$. Thus $\{y'_\alpha\}_{\alpha \in \Delta} \subseteq L(x) \cap U(a)$.

Let $x'_{\alpha} = \wedge_{\beta \geq \alpha} y'_\beta$ for each $\alpha \in \Delta$. Clearly $\{x'_\alpha\}_{\alpha \in \Delta}$ is an increasing net contained in $L(x)$. Let $U$ be an open set containing $x$. There exists a closed sublattice $V \subseteq U$ such that $x \in V^\circ$ [6, Theorem 5]. Since $\{y'_\alpha\}_{\alpha \in \Delta}$ converges to $x$, there exists $\beta \in \Delta$ such that $\alpha \geq \beta$ implies $y'_\alpha \in V^\circ$. Hence $x'_{\beta} = \wedge_{\gamma \geq \beta} y'_\gamma \in V$ for all $\gamma \geq \beta$. Thus $\{x'_\alpha\}_{\alpha \in \Delta}$ converges to $x$. That $\{x'_\alpha\}_{\alpha \in \Delta} \subseteq U(a)$ follows just as did the fact that $\{y'_\alpha\}_{\alpha \in \Delta} \subseteq U(a)$. Therefore $\{x'_\alpha\}_{\alpha \in \Delta}$ is the required net.

For each $\alpha \in \Delta$, the interval $[a, a \vee x'_\alpha]$ has breadth less than $n$ [7, Lemma 1.1]. Since $\{x'_\alpha\}_{\alpha \in \Delta}$ is increasing, $\{[a, a \vee x'_\alpha]\}_{\alpha \in \Delta}$ is a chain. Therefore $\bigcup_{\alpha \in \Delta} [a, a \vee x'_\alpha]$ has breadth less than $n$; consequently the breadth of $(\bigcup_{\alpha \in \Delta} [a, a \vee x'_\alpha])^* \leq n$. Since the interval $[a, a \vee x'_\alpha] = (a \vee x'_\alpha) \wedge M(a)$, it follows that $[a, x] = (\bigcup_{\alpha \in \Delta} [a, a \vee x'_\alpha])^*$ [3, Theorem 3]. Thus $[a, x]$ has breadth less than $n$, and since $[a, x]$ is a compact, connected topological lattice, its breadth and its codimension are equal.

The semilattice $F(M(a))$ is a Lawson semilattice, i.e. has a neighborhood basis of subsemilattices [5, Theorem 1.1]. A. Y. W. Lau has shown that any compact, connected, locally connected Lawson semilattice $S$ contains a point $x$ for which the codimension of $S$ and the codimension of $L(x)$ are equal [4, Lemma 5.2]. Thus for some $x \in F(M(a))$ the codimension of $[a, x]$ and the codimension of $F(M(a))$ are equal.

Corollary. Let $L$ be a compact connected topological lattice of finite codimension $n$. If $a, b \in L$ and $a < b$, then the codimension of $F([a, b])$ is less than $n$. Thus $L$ has a basis of open sets whose boundaries have codimension less than $n$.  

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Proof. If $a=0$ or $b=1$, then $[a, b] = L(b)$ or $[a, b] = M(a)$. Thus we may assume $a \neq 0$ and $b \neq 1$. It is easy to see that $F([a, b])$ is a closed subset of $F(M(a)) \cup F(L(b))$. By the theorem each of these sets has co-dimension less than $n$. Thus the codimension of $F([a, b])$ is less than $n$.

Since $L$ has a basis of neighborhoods at each point consisting of intervals $[a, b]$ [6, Theorem 5], the collection of interiors of these intervals is the desired basis.

References


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