A SPECTRAL SEQUENCE FOR THE INTERSECTION OF SUBSPACE PAIRS

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Abstract. A general-homology spectral sequence that generalizes the Mayer-Vietoris exact sequence is established between the intersection of a family of subspace pairs and the system of partial unions of the family. The basis of the construction is a topological analogue of the "bar construction" of homological algebra.

We shall show here that a finite family \( \mathcal{P} = \{(X_i, A_i) | i \in I\} \) of subspace pairs\(^1\) in a space \( X \) have, for each general homology theory \( h_* \), a spectral sequence

\[
E_1^{p,q} \cong \bigoplus_{Ns=n} h_j \left( \bigcup_{s} X_i, \bigcup_{s} A_i \right) \Rightarrow h_{j-n} \left( \bigcap_{I} X_i, \bigcap_{I} A_i \right)
\]

\((s \subseteq I), Ns being (number of members in s) - 1\). This is just the spectral sequence of a cover with the roles of union and intersection interchanged. Its connection with the Mayer-Vietoris sequence will be examined below, and we shall derive from it the spectral sequence of the homology sheaf of \( X \).

Construction of (a). Start with any finite set \( U (= the universe) that contains \( I \) as a subset, and define (using \( T \) to denote the based unit interval, while \( J \) to \( \gamma \) \( \gamma J \) \( \gamma J \) for based spaces \( Y \) and finite sets \( J \))

\[
\begin{align*}
\nabla_s &= \bigwedge_{s} \partial T \wedge \bigwedge_{U-s} T \quad (s \subseteq U), \\
K &= \bigcup_{a \in U} \nabla\{a\} = \partial \nabla \emptyset, \\
C &= \{\emptyset\} \cup \bigcup_{a \in U-I} \nabla\{a\} = \bigwedge_{I} T \wedge \partial \bigwedge_{U-I} T, \\
M &= X \times C \cup \bigcup_{s \subseteq I} A_i \times \nabla s, \\
L^n &= M \cup \bigcup_{s \subseteq I; Ns \geq n} X_i \times \nabla s \quad (n \in \mathbb{Z}) \\
L &= L^0 = X \times C \cup \bigcup_{s \subseteq I} X_i \times \nabla s.
\end{align*}
\]

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\(^1\) Assumed to be subcomplex pairs under some CW complex structure on \( X \).

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We require $I \neq \emptyset$ but permit $U = I$. Caution. Here $\partial$ is used in the context of based spaces, so $\partial \partial T = \{ \ast \}$, not $\emptyset$. The formulas for the spectral sequence are as follows (as in [1, p. 108 ff.] for any filtered space):

$$E^n_{r,j} = \frac{\text{Im}[h_p(L^n, L^{n+r}) \to h_p(L^{n-r+1}, L^{n+r})]}{\text{Im}[h_p(L^{n+1}, L^{n+r}) \to h_p(L^{n-r+1}, L^{n+r})]} \quad (n, j \in \mathbb{Z}; \ r = 1, 2, \cdots)$$

where $p = j - n + NU$ and maps are induced by inclusion,

(c) $d^n_{r,j} = \text{homomorphism } E^n_{r,j} \to E^n_{r,j+r-1}$ induced by $\partial$, for the triple $(L^n, L^{n+r}, L^{n+2r})$,

$$u^n_{r,j} = \text{isomorphism } E^n_{r+1,j} \to E^n_{r,j+r}$ induced by $h_p(\ast)$ for the inclusion $(L^n, L^{n+r+1}) \subset (L^n, L^{n+r})$.

Thus, $E^n_{\infty,j} = E^n_{r,j}$ for large $r$, $= F^nG_{j-n}/F^{n+1}G_{j-n}$, where

$$G_q = h_{q+NU}(L, M) \quad (q \in \mathbb{Z}),$$
$$F^nG_q = \text{Im}[h_{q+NU}(L^n, M) \to h_{q+NU}(L, M)].$$

(Note that $G_\ast = F^0G_\ast \supset F^1G_\ast \supset \cdots \supset F^{N+1}G_\ast = \{0\}$.)

Define also, for each $a \in U$ and $s \subseteq U$ containing $a$,

$$v(a): h_q(\nabla s, \partial \nabla s) \xrightarrow{\text{exclusion}} h_q(\partial \nabla s', \partial \nabla s' - \nabla^\#s) \quad (s' = s - \{a\}),$$
$$\mu(a): h_{q+1}(\nabla s', \partial \nabla s') \xrightarrow{\delta_{q+1}} h_q(\partial \nabla s', \partial \nabla s' - \nabla^\#s),$$
$$\sigma(a) = \mu(a)^{-1}v(a),$$

where $\nabla^\#(\cdot) = \nabla(\cdot) - \partial \nabla(\cdot)$ and $h_\ast = \text{any general homology theory}, q \text{ any integer}$. For distinct $a_1, \cdots, a_k \in s \subseteq U \ (k \geq 1)$, denote $s - \{a_1, \cdots, a_k\}$ as $s''$ and define

(d, Cont’d.) $\sigma(a_\ast): h_q(\nabla s', \partial \nabla s') \xrightarrow{\text{exclusion}} h_{q+k}(\nabla s'', \partial \nabla s'')$ as $\sigma(a_1) \cdots \sigma(a_k)\sigma(a_\ast)$, where $a_\ast$ means $(a_1, \cdots, a_n)$. $\sigma(a_\ast)$ is alternating, because, for any permutation of $a_\ast$, the corresponding coordinate transformation of $\nabla s''$ permutes the factors of $\sigma(a_\ast)$ in the same way.

Now let $a_\ast = (a_0, \cdots, a_{NU})$ be a choice of numbering of $U$, and for each nonempty subset $s \subseteq I$ let $i_\ast^s = (i_0^s, \cdots, i_N^s)$ be a choice of numbering of $s$. They, together with $\sigma$ and Lemmas 1, 2 below, determine two
isomorphisms:

\[ h_q \left( \bigcap_I X_i, \bigcap_I A_i \right) \]

\[ \cong (-1)^{\sum N_3 \sigma(i_n)} \]

\[ h_{q+NU+1} \left( \left( \bigcap_I X_i, \bigcap_I A_i \right) \times (\nabla \emptyset, \nabla \emptyset) \right) \]

(Lemma 1) \[ \cong h_{q+NU+1} \text{ of the homology theory} \]

\[ h_* = h_* \left((\bigcap_I X_i, \bigcap_I A_i) \times (-)\right) \]

\[ \Psi_j \lambda_{\emptyset} \]

\[ h_{q+NU} \left( \left( \bigcap_I X_i, \bigcap_I A_i \right) \times (K, C) \right) \]

(Lemma 1) \[ \cong h_{q+NU} \subset \]

\[ h_{q+NU} (L, M) \]

\[ (e) \]

\[ \bigoplus_{N_S = n, S \subseteq I} h_j \left( \bigcup_S X_i, \bigcup_S A_i \right) \]

\[ \cong (-1)^{\sum N_3 \sigma(i_n)} \]

\[ h_p \left( \left( \bigcup_S X_i, \bigcup_S A_i \right) \times (\nabla S, \nabla S) \right) \]

(Lemma 2) \[ \cong \sum h_p \subset \]

\[ h_p (L^n, L^{n+1}) \]

\[ \sum \lambda_{\emptyset} \]

\[ \bigoplus_{N_S = n, S \subseteq I} h_* (L^n, L^{n+1}) \]

\[ \bigoplus \]

\[ E_{1; j} \]

\[ (q, n, j \in \mathbb{Z}) \text{ which combine with formulas (c) to give the formula (a).} \]

**Lemma 1.** \( \lambda_{\emptyset} \) is an isomorphism.

**Proof.** (Referring to (e).) \( \partial_{q+NU+1} \) is bijective by contractibility of \( (\nabla \emptyset, C) = \bigwedge_T T \wedge (\bigwedge_{U \to T} T, \partial \bigwedge_{U \to T} T) \). For bijectivity of \( h_{q+NU} \subset \) it suffices by the Five Lemma to consider the case \( A_i = \emptyset \) (all \( i \in I \)). Define

\[ L^n = X \times C \cup \bigcup_{S \subseteq I; N_S \equiv n} \left[ \bigcap S X_i \times \bigcup_S \nabla \{i\} \right] \]

\( (n \in \mathbb{Z}) \).

\[ h_\ast (L^n, L^{n+1}) \cong \bigoplus_{N_S = n} h_\ast \left( \left( \bigcap S X_i, \bigcup_S X_i \cap \bigcap_S X_i \right) \times (C \cup \bigcup_S \nabla \{i\}, C) \right) \]
by additivity of homology, \( \cong \{0\} \) by contractibility of
\[ (C \cup \bigcup_i \nabla \{i\}, C) = \bigwedge_{i-s} T \wedge \left( \partial_{s \cup (U-I)} T, \bigwedge_{i-s} T \wedge \partial_{U-I} T \right), \]
assuming \( 0 \leq n < NI \). \( h_* (\bigcap_i X_i \times (K, C)) \cong h_* (L^{(NI)}, X \times C) \) by excision, \( \cong h_* (L^{(NI-1)}, X \times C) \cong \cdots \cong h_* (L^0, X \times C) = h_* (L, X \times C) \) by exactness using above \( \{0\} \).

**Lemma 2.** \( \sum \lambda_s \) is an isomorphism.

**Proof.** Additivity of homology. \( \square \)

**Comparison with the Mayer-Vietoris sequence.** Since (a) relates the various unions of the pairs \( \mathcal{P} \) to their intersection it brings to mind the Mayer-Vietoris sequence. (a) is in fact a generalization of the latter, as we shall now show. (The Mayer-Vietoris sequence is the \( NI=1 \) case of \( \epsilon, \delta^0, \beta \) below.)

For any \( n \in \mathbb{Z} \) let \( S_n(I) = \{ i_* = (i_0, \cdots, i_n) | i_0, \cdots, i_n \in I \} \), which is to entail that \( S_n(I) = \emptyset \) for negative \( n \). Using \( X_{i_*} \) to mean \( X_{i_0} \cup \cdots \cup X_{i_n} \), define \( C^n(\mathcal{P}; h_j(\bigcup \cdot)) \) \((j \in \mathbb{Z}) = \) subgroup of \( \prod_{i_* \in S_n(I)} h_j (X_{i_*}, A_{i_*}) \) consisting of alternating members \( \xi = \{ \xi_{i_*} | i_* \in S_n(I) \} \) for which \( \xi_{i_*} = 0 \) whenever two or more of \( i_0, \cdots, i_n \) are equal, and note that
\[ C^n(\mathcal{P}; h_j(\bigcup \cdot)) \cong \bigoplus_{N_s=n; s \subseteq I} h_j \left( \bigcup_{i_* \subseteq I} X_{i_*} \cup A_{i_*} \right) \]
under the correspondence \( \xi \mapsto \{ \xi_{i_*} | N_s=n, s \subseteq I \} \). Denote by \( \Phi^n_j \) the composite of \( \varphi^n_j \) with this isomorphism.

**Lemma 3.** The following diagram commutes:
\[
\begin{array}{ccccccc}
E^0_{1;j} & \rightarrow E^1_{1;j} & \rightarrow & \cdots \\
\downarrow d^0_{1;j} & & & \downarrow d^1_{1;j} & & & \downarrow d^1 \\
E^1_{0;j} & \rightarrow E^0_{1;j} & \rightarrow & \cdots
\end{array}
\]
where
\[ d^0_{1;j} = d^1_{1;j} = d^1 \]
and
\[ \delta^n(\xi)_{i_*} = \sum_{0 \leq k \leq n+1} (-1)^k \xi_{i_*+k} | X_{i_*} \cup A_{i_*} \]
\[ (n \in \mathbb{Z}, \xi \in C^n(\mathcal{P}; h_j(\bigcup \cdot)), i_* \in S_{n+1}(I)) \]
i_* (k) being \( (i_0, \cdots, i_{k-1}, i_{k+1}, \cdots, i_{n+1}) \).
Proof. For the square involving $\delta^n$ for some $n$, $0 \leq n \leq NI$, we consider an arbitrary element of $C^n(\mathcal{P}; h_j(\cup \cdot))$ of the form $\chi(i_*; \theta)$ defined as follows: $i_*$ is a numbering of a subset $s \subseteq I$ with $N_s = n$, $\theta$ belongs to $h_j(\bigcup X_i, \bigcup A_i)$, and $\chi(i_*; \theta)i_* = \pm \theta$ or 0, depending upon whether $i'_*$ is an even or odd permutation of $i_*$ or not a permutation of $i_*$, respectively. $\delta^n\chi(i_*; \theta) = \sum_{i \in I-s} \chi(ii_*; \theta|x_{ii_*}. A_{ii_*})$. In the commutative diagram

$$h_p(\nabla s, \partial \nabla s)^{\{u(i)\}} \xrightarrow{\oplus h_{p-1}(\partial \nabla s, \partial \nabla s - \nabla^#(s \cup i))} \oplus h_{p-1}(\nabla(s \cup i), \partial \nabla(s \cup i))$$

assume $h_* = h_*((X_{i_*}, A_{i_*}) \times \cdot)$, that each $R^{i_*}_{i_*}$ is induced by the appropriate inclusion, and each sum or union is taken over $\{i|i \in I-s\}$. We have that $d^n_1 : h_* = h_{p-1}(\cdot) \partial_p$, etc., $= \sum \lambda_{i \cup i} R^{i_*}_{i_*} \sigma(i)^{-1}$. Therefore,

$$(-1)^{nU} d^n_1 \Phi^n_{ij} \chi(i_*; \theta) = d^n_1 : \lambda_{i_*} \sigma(i_*)^{-1} \sigma(a_*) \theta$$

$$= \sum \lambda_{i \cup i} R_{i_*}^{i_*} \sigma(i)^{-1} \sigma(a_*) \theta$$

$$= \sum \lambda_{i \cup i} \sigma(i)^{-1} \sigma(a_*) (\theta|x_{ii_*}. A_{i_*})$$

$$= (-1)^{nU} \cdot \Phi^{n-1} \delta^n \chi(i_*; \theta).$$

We have thus proved the square commutative, since the $\chi(i_*; \theta)$’s generate $C^n(\mathcal{P}; h_j(\cup \cdot))$. For the square involving $\epsilon$, the same argument works with $h_*$ redefined as $h_*((\bigcap X_i, \bigcap A_i) \times \cdot)$, $s$ replaced by $\varnothing$, and $d^n_1$ replaced by $\kappa_\varnothing$. □

Lemma 4. If $A_i = \varnothing$ (all $i \in I$), the following diagram commutes:

$$C^{NI}(\mathcal{P}; h_j(\cup \cdot)) \xrightarrow{\beta} h_{j-NI}(\bigcap X_i) \xrightarrow{\Phi_j^{NI}} h_{j-NI}(\bigcap X_i)$$

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where $\beta$ has the formula

$$C^{NI}(\mathcal{P}; h_j(\cup \cdot)) \cong h_j\left(\bigcup_i X_i \right) \to h_j\left(\bigcup_i X_i, \bigcup_{j < NI} X_{(j)}\right)$$

$$\cong \text{excision} \quad h_j\left(X_{(NI)}, \bigcup_{j < NI} X_{(j)} \cap X_{(NI)}\right)$$

$$\cong (-1)^{(NI)\beta_1 \beta_2 \cdots \beta_{NI}} \quad h_{j-\text{NI}}\left(\bigcap_i X_i\right) \quad (X_{(j)} = X_{ij}),$$

each $\beta_k$ ($1 \leq k \leq NI$) being the composite

$$h_{j-\text{NI}+k}\left(\bigcap_{j \geq k} X_{(j)}, \bigcup_{j < k} X_{(j)} \cap \bigcap_{j \geq k} X_{(j)}\right)$$

$$\downarrow \delta_{j-\text{NI}+k}$$

$$h_{j-\text{NI}+k-1}\left(\bigcup_{j < k} X_{(j)} \cap \bigcap_{j \geq k} X_{(j)}, \bigcup_{j < k-1} X_{(j)} \cap \bigcap_{j \geq k-1} X_{(j)}\right)$$

$$\cong \text{excision} \quad h_{j-\text{NI}+k-1}\left(\bigcup_{j \geq k-1} X_{(j)}, \bigcup_{j < k-1} X_{(j)} \cap \bigcap_{j \geq k-1} X_{(j)}\right).$$

**Proof.** Omitted. Consists of comparing each $\beta_k$ with the appropriate form of $\sigma(i^T_j)^{-1}$ in one large commutative diagram. \(\square\)

**Independence from $U$.** Let $U^+ = U \oplus \{a\}$ for some point $a$ apart from $U$, and indicate by a superscript $+$ the $U^+$-version of each of the notions (b)–(e). To prove that the choice of $U$ is immaterial it suffices to prove $(c) \cong (c^+)$, $(e) \cong (e^+)$. We therefore define an isomorphism

$$l^{n,m}: h_p(L^n, L^m) \to h_{p^+}(L^{n^+}, L^{m^+})$$

as follows, for $n, j$ as in (c) and $m \geq n$:

$$h_p(L^n, L^m) = h_p^{(n,m)}\left(\partial \wedge T, \{\ast\}\right)$$

$$\cong l^{p+1}\left(\partial \wedge T, \{\ast\}\right)$$

Here $h_p^{(n,m)}$ is the general homology theory of based compact pairs $(Y, B)$ with formula $h_p^{(n,m)}(Y, B) = h_p(L^n(Y), L^m(Y) \cup L^n(B))$, $L^n(Y)$ being $X \times C_\ast Y \cup \cup_{s \geq 1} \cup_{s \in I} A_s \times \nabla_s Y \cup \cup_{s \in \bar{I}; \bar{N}_s \geq n} [\cup_{s} X_t \times \nabla_s Y]$. Then, $(-1)^{j-n-1}l^{n,n^+}$ induces an isomorphism $E^{r,j}_n \to E^{r,j}_n$ $(n, j \in \mathbb{Z}; r = 1, 2, \cdots)$ that carries $d^{r,j}_n$ into $d^{r,j}_n$, $\Phi^+_j$ (for $r = 1$) into $\Phi^+_j$, etc., as required. We assume that $a^+_r = a^+_r a$. 

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Functoriality. Constructing (a) is more difficult than constructing the spectral sequence of a cover in that the underlying spaces (b) do not depend functorially on \((X, I, \mathcal{P})\). \(U\) has been introduced as a remedy.

We assume that a morphism from \((X, I, \mathcal{P})\) to another such triple \((Y, J, \mathcal{Q})\), \(\mathcal{Q}\) being a finite family \(\{(Y_j, B_j)\}_{j \in J}\) of subspace pairs in a space \(Y\), is a map \(f: X \to Y\) of spaces together with a map \(\pi: J \to I\) of sets such that \((fX_{x_j}, fA_{x_j}) \subset (Y_j, B_j) \quad (j \in J)\). Evidently \(C^n(\mathcal{Q}; h_q(\bigcup \cdot))\) \((n, q \in \mathbb{Z})\) depends functorially on \((X, I, \mathcal{P})\) if \((f; \pi)\) is regarded as inducing the map \(C^n(f; \pi): C^n(\mathcal{P}; h_q(\bigcup \cdot)) \to C^n(\mathcal{Q}; h_q(\bigcup \cdot))\) with the formula \((C^n(f; \pi)\xi)_j = h_q(f; \pi)\xi_j \quad (\xi \in C^n(\mathcal{P}; h_q(\bigcup \cdot)), j \in S_n(J))\), \(h_q(f; \pi)\xi_j\) being the homomorphism \(h_q(X_{x_j}, A_{x_j}) \to h_q(Y_{y_j}, B_{y_j})\) induced by \(f|_{X_{x_j}}\). Similarly, \(h_*(\bigcap I X_i, \bigcap I A_i)\) is functorial, the induced map to be denoted \(h_*(f; \bigcap)\).

Let primes signify the \((Y, J, \mathcal{Q})\)-version of the notions (b)-(e). To show that \((c), \Psi_*, \Phi_*\) depend functorially on \((X, I, \mathcal{P})\), we need only produce a homomorphism of \((c)\) to \((c')\) which, when considered along with \(C^*\) and \(h_*(f; \bigcap)\), maps \(\Psi_*, \Phi_*\) to \(\Psi'_*, \Phi'_*\) respectively. It is easy to see that this map of \((c)\) to \((c')\) is a fortiori unique and functorially dependent on the morphism \((f; \pi)\).

We start by assuming \(U' = U \supset I \oplus J\). Define \(\omega: \bigwedge_U T \to \bigwedge_V T\) to be the involution \(\bigwedge_{U-\{\pi J \cup J\}} \omega_i\), where, for each \(i \in \pi J\),

\[
\omega_i\left(t_i \wedge \bigwedge_{\pi-1(i)} t_j\right) = m \wedge \bigwedge_{\pi-1(i)} (t_j t_j/m)
\]

\((t_i, t_j \in T\) for \(j \in \pi^{-1}(i))\), \(m\) being \(\text{Max}_{\pi^{-1}(i)} t_j\). It is easily shown that \(\omega \vee \{i\} = \bigcup_{\pi^{-1}(i)} \vee \{j\}\) for \(i \in \pi J\), while \(\omega \vee \{i\} \subset C'\) for \(i \in I-\pi J\). The consequence is \((f \times \omega)L^n \subset L^n (n \in \mathbb{Z})\), with an induced homomorphism \(L^n, m: h_p(L^n, L^m) \to h_p(L'^n, L'^m) (m \geq n)\). The map \((-1)^{\text{number of members in } \pi J}\) \(L^n, n+r\) induces the required \(E^r_{n,j} \to E^r_{n,j} (n, j \in \mathbb{Z}; r=1, 2, \cdots)\). (The power of \((-1)^i\) is the degree of \(\omega_i\).

The homology sheaf. Let \(\mathcal{P} = \{(X, A \cup (X- U^i))| i \in I\} = \mathcal{P}_\mathcal{U}\) for some finite open cover \(\mathcal{U} = \{U^i|i \in I\}\) of \(X\), \(A\) being some subspace. Evidently

\[
h_*\left(\bigcap_I X_i, \bigcap_I A_i\right) = h_*(X, A),
\]

\[
C^*(\mathcal{P}_\mathcal{U}; h_*(\bigcup \cdot)) = C^*(\mathcal{U}; h^X, A),
\]

where \(h^X, A\) is the graded presheaf \(\{h_*(X, A \cup (X- \emptyset))| \text{open } \emptyset \subset X\}\). Thus, we obtain a spectral sequence

\[
E^n_{2,j} \cong H^n(\mathcal{U}; h^X, A) \Rightarrow h_{j-n}(X, A).
\]
For $X$ compact, the direct limit of $(f)$, as $\mathcal{U}$ is refined, is a spectral sequence
\[(g)\quad E^n_{0,j} \cong H^n(X; \mathcal{H}^X_{j,A}) \Rightarrow h_{j-n}(X, A),\]
where $\mathcal{H}^X_{j,A}$ is the induced sheaf of $h^X$.$A$. As $A$ approximates an open set $V$ from within, the direct limit of $(g)$ is
\[(h)\quad E^n_{0,j} \cong H^n(X, V; \mathcal{H}^X_{j}) \Rightarrow h_{j-n}(X, V).\]

$\mathcal{H}^X_*$ is called the homology sheaf of $X$. If $\mathcal{H}^X_j \cong \{0\}$ except for $j=j_0$ (\(=\) some integer), e.g., if $X$ is a $j_0$-manifold and $h_*$ is standard, then $(h)$ collapses to a family of isomorphisms
\[H^n(X, V; \mathcal{H}^X_{j_0}) \cong h_{j_0-n}(X, V) \quad (n \in \mathbb{Z}).\]
(Compare to [2].)

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