A SPECTRAL SEQUENCE FOR THE INTERSECTION OF SUBSPACE PAIRS

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Abstract. A general-homology spectral sequence that generalizes the Mayer-Vietoris exact sequence is established between the intersection of a family of subspace pairs and the system of partial unions of the family. The basis of the construction is a topological analogue of the “bar construction” of homological algebra.

We shall show here that a finite family \( \mathcal{P} = \{(X_i, A_i) | i \in I\} \) of subspace pairs in a space \( X \) have, for each general homology theory \( h_* \), a spectral sequence

\[
E^n_{i,j} \cong \bigoplus_{N_s = n} h_j \left( \bigcup_{s \in I} X_i, \bigcup_{s \in I} A_i \right) \Rightarrow h_{j-n} \left( \bigcap_{i \in I} X_i, \bigcap_{i \in I} A_i \right)
\]

\((s \subseteq I), N_s \) being (number of members in \( s \))\(-1\). This is just the spectral sequence of a cover with the roles of union and intersection interchanged. Its connection with the Mayer-Vietoris sequence will be examined below, and we shall derive from it the spectral sequence of the homology sheaf of \( X \).

Construction of (a). Start with any finite set \( U \) (= the universe), and define (using \( T \) to denote the based unit interval, while \( \bigcup J \) for based spaces \( Y \) and finite sets \( J \))

\[
\nabla_s = \bigwedge_{s \subseteq I} T \wedge T
\]

\[
K = \bigcup_{a \in U} \nabla \{a\} = \partial \nabla \emptyset,
\]

\[
C = \{\ast\} \cup \bigcup_{a \in U-I} \nabla \{a\} = \bigwedge_{I} T \wedge \partial \bigwedge_{U-I} T,
\]

\[
M = X \times C \cup \bigcup_{s \subseteq I} \bigcup_{s \in I} X_i \times \nabla_{s},
\]

\[
L^n = M \cup \bigcup_{s \subseteq I; N_s = n} \bigcup_{s \subseteq I} X_i \times \nabla_{s},
\]

\[
L = L^0 = X \times C \cup \bigcup_{s \subseteq I} \bigcup_{s \subseteq I} X_i \times \nabla_{s}.
\]

Received by the editors February 16, 1973.

AMS (MOS) subject classifications (1970). Primary 55H05, 55H25; Secondary 55C05.

Key words and phrases. Spectral sequence, general homology, general cohomology, cover, duality.

1 Assumed to be subcomplex pairs under some CW complex structure on \( X \).
We require \( I \neq \emptyset \) but permit \( U = I \). Caution. Here \( \partial \) is used in the context of based spaces, so \( \partial \partial T = \{ \ast \} \), not \( \emptyset \). The formulas for the spectral sequence are as follows (as in [1, p. 108 ff.] for any filtered space):

\[
E_{r,j}^n = \frac{\text{Im}[h_p(L^n, L^{n+r}) \to h_p(L^{n-r+1}, L^{n+r})]}{\text{Im}[h_p(L^{n+1}, L^{n+r}) \to h_p(L^{n-r+1}, L^{n+r})]} \quad (n, j \in \mathbb{Z}; \ r = 1, 2, \ldots)
\]

where \( p = j - n + NU \) and maps are induced by inclusion, \( \partial \).

(c) \( d_{r,j}^n = \text{homomorphism } E_{r,j}^n \to E_{r,j+r-1}^n \) induced by \( \partial \), for the triple \( (L^n, L^{n+r}, L^{n+2r}) \),

\[
u_{r,j}^n = \text{isomorphism } E_{r+1,j}^n \to E_{r,j}^n \]

induced by \( h_p(\ast) \) for the inclusion \( (L^n, L^{n+r+1}) \subset (L^n, L^{n+r}) \).

Thus, \( E_{\infty,j}^n = E_{r,j}^n \) for large \( r \), \( = F^n G_{j-n} \cap F^{n+1} G_{j-n} \), where

\[
G_q = h_{q+NU}(L, M) \quad (q \in \mathbb{Z}),
\]

\[
F^n G_q = \text{Im}[h_{q+NU}(L^n, M) \to h_{q+NU}(L, M)].
\]

(Note that \( G_\ast = F^0 G_\ast \supset F^1 G_\ast \supset \cdots \supset F^{NI+1} G_\ast = \{0\} \).

Define also, for each \( a \in U \) and \( s \subset U \) containing \( a \),

\[
u(a): h_q(\nabla s, \partial \nabla s) \xrightarrow{\text{excision}} h_q(\partial \nabla s', \partial \nabla s' - \nabla^\# s) (s' = s - \{a\}),
\]

\[
\mu(a): h_{q+1}(\nabla s', \partial \nabla s') \xrightarrow{\text{excision}} h_q(\partial \nabla s', \partial \nabla s' - \nabla^\# s),
\]

\[
\sigma(a) = \mu(a)^{-1} \nu(a),
\]

where \( \nabla^\#(\cdot) = \nabla(\cdot) - \partial \nabla(\cdot) \) and \( h_\ast = \text{any general homology theory} \), \( q \) any integer. For distinct \( a_1, \ldots, a_k \in s \subset U \) \( (k \geq 1) \), denote \( s - \{a_1, \ldots, a_k\} \) as \( s'' \) and define

(d, Cont'd.) \( \sigma(a_\ast): h_q(\nabla s, \partial \nabla s) \xrightarrow{\text{excision}} h_{q+k}(\nabla s'', \partial \nabla s'') \)

as \( \sigma(a_1) \cdots \sigma(a_k)(a_\ast) \), where \( a_\ast \) means \( (a_1, \ldots, a_n) \). \( \sigma(a_\ast) \) is alternating, because, for any permutation of \( a_\ast \), the corresponding coordinate transformation of \( \nabla s'' \) permutes the factors of \( \sigma(a_\ast) \) in the same way.

Now let \( a_\ast = (a_0, \ldots, a_{NU}) \) be a choice of numbering of \( U \), and for each nonempty subset \( s \subset I \) let \( i_\ast = (i_0, \ldots, i_N) \) be a choice of numbering of \( s \). They, together with \( \sigma \) and Lemmas 1, 2 below, determine two
isomorphisms:

\[ h_q \left( \bigcap_i X_i, \bigcap_i A_i \right) \]

\[ \cong (-1)^{NU+q}(\sigma(a_*)) \]

(lemma 1)

\[ \lambda_{q+NU} \left( \left( \bigcap_i X_i, \bigcap_i A_i \right) \times (K, C) \right) \]

(lemma 1)

\[ h_{q+NU}(L, M) \]

(\(q, n, j \in \mathbb{Z}\)) which combine with formulas (c) to give the formula (a).

**Lemma 1.** \(\lambda_{q}\) is an isomorphism.

**Proof.** (Referring to (e.) \(\partial_{q+NU+1}\) is bijective by contractibility of \((\nabla \varnothing, C) = \bigwedge T^*(\bigwedge U - T, T, \partial (\bigwedge U - T))\). For bijectivity of \(h_{q+NU}(\subset)\) it suffices by the Five Lemma to consider the case \(A_i = \varnothing\) (all \(i \in I\)). Define

\[ L^{(n)} = X \times C \cup \bigcup_{s \subseteq I; N_s \equiv n} \left[ \bigcap_i X_i \times \bigcup_i \nabla\{i\} \right] \quad (n \in \mathbb{Z}). \]

\[ h_* (L^{(n)}, L^{(n+1)}) \cong \bigoplus_{N_s = n} h_* \left( \left( \bigcap_i X_i, \bigcap_i X_i \cap \bigcup \nabla\{i\} \right) \times (C \cup \bigcup_i \nabla\{i\}, C) \right) \]
by additivity of homology, $\cong \{0\}$ by contractibility of
\[
\left( C \cup \bigcup_z \nabla \{i\}, C \right) = \bigwedge_{I=0}^{n} T \wedge \left( \partial_{s \cup U(I)} T, \bigwedge_z T \wedge \partial_{U(I)} T \right),
\]
assuming $0 \leq n < NI$. $h_{\ast}(\bigcap_I X_i \times(K, C)) \cong h_{\ast}(L^{(NI)}, X \times C)$ by excision, $h_{\ast}(L^{(NI-1)}, X \times C) \cong \cdots \cong h_{\ast}(L^{(0)}, X \times C) = h_{\ast}(L, X \times C)$ by exactness using above $\{0\}$. □

**Lemma 2.** $\sum \lambda_s$ is an isomorphism.

**Proof.** Additivity of homology. □

**Comparison with the Mayer-Vietoris sequence.** Since (a) relates the various unions of the pairs $\mathcal{P}$ to their intersection it brings to mind the Mayer-Vietoris sequence. (a) is in fact a generalization of the latter, as we shall now show. (The Mayer-Vietoris sequence is the $NI=1$ case of $\varepsilon, \delta_{0}, \beta$ below.)

For any $n \in \mathbb{Z}$ let $S_n(I) = \{i_\ast = (i_0, \ldots, i_n) | i_0, \ldots, i_n \in I\}$, which is to entail that $S_n(I) = \emptyset$ for negative $n$. Using $X_{i_\ast}$ to mean $X_{i_0} \cup \cdots \cup X_{i_n}$, define
\[
C^n(\mathcal{P}; h_j(\bigcup \cdot)) \quad (j \in \mathbb{Z}) = \text{subgroup of } \bigcup_{i_\ast \in S_n(I)} h_{\ast}(X_{i_\ast}, A_{i_\ast})
\]
consisting of alternating members $i_\ast$ for which $|i_\ast| = 0$ whenever two or more of $i_0, \ldots, i_n$ are equal, and note that
\[
C^n(\mathcal{P}; h_j(\bigcup \cdot)) \cong \bigoplus_{N=0; s \subset I} h_{\ast}\left( \bigcup_z X_i, \bigcup_z A_i \right)
\]
deriving from the correspondence $\xi \mapsto \{\xi^i|\ast \in S_n(I)\}$. Denote by $\Phi_{\ast}^n$ the composite of $\varphi^n$ with this isomorphism.

**Lemma 3.** The following diagram commutes:
\[
\begin{array}{cccccccccccccccc}
\end{array}
\]

where
\[
\delta^n(\xi)^{i_0} = \sum_{0 \leq k \leq n+1} (-1)^k \xi^{i_0(k)} X^{i_{k-1}, \ldots, i_{n+1}} (n \in \mathbb{Z}, \xi \in C^n(\mathcal{P}; h_j(\bigcup \cdot)), i_\ast \in S_{n+1}(I))
\]

and
\[
i_\ast(k) \text{ being } (i_0, \ldots, i_{k-1}, i_{k+1}, \ldots, i_{n+1}).
\]
Proof. For the square involving $\delta^n$ for some $n$, $0 \leq n \leq N_I$, we consider an arbitrary element of $C^n(\mathcal{P}; h_I(\bigcup \cdot))$ of the form $\chi(t_{i*}; \theta)$ defined as follows: $i_*$ is a numbering of a subset $s \subseteq I$ with $N_s = n$, $\theta$ belongs to $h_I(\bigcup_s X_i, \bigcup_s A_i)$, and $\chi(t_{i*}; \theta) = \pm \theta$ or 0, depending upon whether $i_*$ is an even or odd permutation of $i_*$ or not a permutation of $i_*$, respectively. $\delta^n\chi(t_{i*}; \theta) = \sum_{i \in I-s} \chi(t_{i*}; \theta|X_{i*} \cdot A_{i*})$. In the commutative diagram

$$h_{p-1}(\nabla S, \partial \nabla S) \xrightarrow{(\mu_{v(i)})} h_{p-1}(\partial \nabla S, \partial \nabla S - \nabla#(s \cup i)) \xrightarrow{\oplus h_{p-1}(\nabla(s \cup i), \partial \nabla(s \cup i))} \oplus h_{p-1}(\nabla(s \cup i)),$$

$$h_{p-1}(L^n, L^{n+1}) \xrightarrow{\delta_p} h_{p-1}(\partial \nabla S, \partial \nabla S - \bigcup \nabla#(s \cup i)) \xrightarrow{\oplus \sum_{i \subseteq} h_{p-1}(i)} \oplus h_{p-1}(\nabla(s \cup i), \partial \nabla(s \cup i)),$$

$$h_{p-1}(L^{n+1}, L^{n+2}) \xrightarrow{\lambda_{i*}} h_{p-1}(X_{i*}, A_{i*}) \times (\nabla(s \cup i), \partial \nabla(s \cup i)),$$

assume $h_* = h_*(X_{i*}, A_{i*}) \times (\cdot)$, that each $R^{i*}_{i*}$ is induced by the appropriate inclusion, and each sum or union is taken over $\{i \in I-s\}$. We have that $d_{i*}^{m-1} = h_{p-1}(-)\partial_p$, etc., $= \sum_{i \subseteq} \lambda_{i*}R^{i*}_{i*}\sigma(i)^{-1}$. Therefore,

$$(-1)^{j \mu N} \Phi_{i*}^{m} \chi(t_{i*}; \theta) = d_{i*}^{m-1}\lambda_{i*}\sigma(i)^{-1}\sigma(a_*)\theta$$

$$= \sum \lambda_{i \cup i} R^{i*}_{i*}\sigma(i)^{-1}\sigma(a_*)\theta$$

$$= \sum \lambda_{i \cup i} \sigma(i)^{-1}\sigma(a_*)\theta$$

$$= (1)^{j \mu N} \Phi_{i*}^{m+1}\delta^n\chi(t_{i*}; \theta).$$

We have thus proved the square commutative, since the $\chi(t_{i*}; \theta)$'s generate $C^n(\mathcal{P}; h_I(\bigcup \cdot))$. For the square involving $e$, the same argument works with $h_*$ redefined as $h_*(\bigcap_I X_i, \bigcap_I A_i) \times (\cdot)$, $s$ replaced by $\emptyset$, and $d_{i*}$ replaced by $\kappa_0$. □

Lemma 4. If $A_i = \emptyset$ (all $i \in I$), the following diagram commutes:

$$C^{NI}(\mathcal{P}; h_I(\bigcup \cdot)) \xrightarrow{\beta} h_{I-NI}(\bigcap X_i) \xrightarrow{\Phi_{I}^{NI}} F^{NI}G_{j-NI} \xrightarrow{\Psi_{j-NI}} E_{j-NI} \xrightarrow{E_{j-NI}} F^{NI+1}G_{j-NI} \xrightarrow{\Psi_{j-NI}} G_{j-NI}$$
where \( \beta \) has the formula
\[
C^{NI}(\mathcal{P}; h_j(\bigcup \cdot )) \cong h_j \left( \bigcup_i X_i \right) \to h_j \left( \bigcup_i X_i, \bigcup_{j < NI} X_{(j)} \right)
\]

\[
\overset{\cong}{\text{exclusion}} \to \quad h_j \left( X_{(NI)}, \bigcup_{j < NI} X_{(j)} \cap X_{(NI)} \right)
\]

\[
\overset{(-1)^{NI}}{\to} \beta_1 \beta_2 \cdots \beta_{NI} \quad h_{j-NI} \left( \bigcap_i X_i \right) \quad (X_{(j)} = X_{ij}),
\]

each \( \beta_k \) \((1 \leq k \leq NI)\) being the composite
\[
h_{j-NI+k} \left( \bigcap_{i \geq k} X_{(j)}, \bigcup_{j < k} X_{(j)} \cap \bigcup_{j \geq k} X_{(j)} \right)
\]

\[
\overset{\delta_{j-NI+k}}{\to} \quad h_{j-NI+k-1} \left( \bigcup_{j < k} X_{(j)} \cap \bigcap_{j \geq k} X_{(j)}, \bigcup_{j < k-1} X_{(j)} \cap \bigcap_{j \geq k-1} X_{(j)} \right)
\]

\[
\overset{\cong}{\text{exclusion}} \quad h_{j-NI+k-1} \left( \bigcup_{j < k-1} X_{(j)}, \bigcup_{j < k-1} X_{(j)} \cap \bigcap_{j \geq k-1} X_{(j)} \right).
\]

**Proof.** Omitted. Consists of comparing each \( \beta_k \) with the appropriate form of \( \sigma(i_j) \) in one large commutative diagram. \( \Box \)

**Independence from \( U \).** Let \( U^+ = U \oplus \{a\} \) for some point \( a \) apart from \( U \), and indicate by a superscript + the \( U^+ \)-version of each of the notions (b)--(e). To prove that the choice of \( U \) is immaterial it suffices to prove (c) \( \cong (c^+) \), (e) \( \cong (e^+) \). We therefore define an isomorphism
\[
l_{n,m} : h_p(L^n, L^m) \to h_{p^+}(L^{+n}, L^{+m})
\]
as follows, for \( n, j \) as in (c) and \( m \geq n \):
\[
h_p(L^n, L^m) = \mathbf{h}^{(n,m)}_p(\partial \wedge T, \{\ast\})
\]

\[
\overset{\gamma_{p+1}}{\cong} \quad \mathbf{h}^{(n,m)}_{p+1}((\wedge T, \partial \wedge T) \overset{\cong}{\longrightarrow} h_{p^+}(L^{+n}, L^{+m}).
\]

Here \( \mathbf{h}^{(n,m)}_p \) is the general homology theory of based compact pairs \((Y, B)\) with formula \( h_p^{(n,m)}(Y, B) = h_p(L^n(Y), L^m(Y) \cup L^n(B)), L^n(Y) \) being \( X \times C \wedge Y \cup \bigcup_{s \subset I} [\bigcup A_s \times \nabla s \wedge Y] \cup \bigcup_{s \subset I \cap N} [\bigcup s (X_i \times \nabla s \wedge Y) \cup \bigcup_{s \subset I \cap N} \bigcup_{s \subset I} [\bigcup A_s \times \nabla s \wedge Y]. \) Then, \( (-1)^{j-1} R_{n,j} \) induces an isomorphism \( E_{r,j}^p \to E_{r,j}^{+p} \) \((n, j \in \mathbb{Z}; r = 1, 2, \cdots)\) that carries \( d_{r,j}^p \) into \( d_{r,j}^{+p}, \Phi_j^n \) (for \( r = 1 \)) into \( \Phi_j^{+n}, \) etc., as required. We assume that \( a^+ = a \ast a \).
Functoriality. Constructing (a) is more difficult than constructing the spectral sequence of a cover in that the underlying spaces (b) do not depend functorially on \((X, I, \mathcal{P})\). \(U\) has been introduced as a remedy.

We assume that a morphism from \((X, I, \mathcal{P})\) to another such triple \((Y, J, \mathcal{Q})\), \(\mathcal{Q}\) being a finite family \(\{(Y_j, B_j)\,|\,j \in J\}\) of subspace pairs in a space \(Y\), is a map \(f: X \to Y\) of spaces together with a map \(\pi: J \to I\) of sets such that \((fX_{x_j}, fA_{x_j}) = (Y_j, B_j)\) \((j \in J)\). Evidently \(C^n(\mathcal{P}; h_q(\bigcup \cdot))\) \((n, q \in \mathbb{Z})\) depends functorially on \((X, I, \mathcal{P})\) if \((f; \pi)\) is regarded as inducing the map \(C^n(f; \pi): C^n(\mathcal{P}; h_q(\bigcup \cdot)) \to C^n(\mathcal{Q}; h_q(\bigcup \cdot))\) with the formula \((C^n(f; \pi)\xi)_i = h_q(f; \pi)^i _i \xi^{ij}\) \((\xi \in C^n(\mathcal{P}; h_q(\bigcup \cdot)), j_\ast \in S_n(J)\), \(h_q(f; \pi)^i _i\) being the homomorphism \(h_q(X_{x_j}, A_{x_j}) \to h_q(Y_{t_j}, B_{t_j})\) induced by \(f|_{x_j}\). Similarly, \(h_\ast(\bigcap X_i, \bigcap A_i)\) is functorial, the induced map to be denoted \(h_\ast(f; \cap)\).

Let primes signify the \((Y, J, \mathcal{Q})\)-version of the notions (b)-(e). To show that (c), \(\Psi_\ast, \Phi_\ast\) depend functorially on \((X, I, \mathcal{P})\), we need only produce a homomorphism of (c) to (c') which, when considered along with \(C^\ast(f; \pi)\) and \(h_\ast(f; \cap)\), maps \(\Psi_\ast, \Phi_\ast\) to \(\Psi'_\ast, \Phi'_\ast\) respectively. It is easy to see that this map of (c) to (c') is a fortiori unique and functorially dependent on the morphism \((f; \pi)\).

We start by assuming \(U' = U \cap J\). Define \(\omega: \bigwedge U \to \bigwedge U\) to be the involution \(\bigwedge_{j \in J} \omega_i\), where, for each \(i \in \pi J,\)

\[
\omega_i \left( t_i \wedge \bigwedge_{\pi^{-1}(i)} t_j \right) = m \wedge \bigwedge_{\pi^{-1}(i)} (t_j t_i/m)
\]

\((t_i, t_j \in T\) for \(j \in \pi^{-1}(i)\), \(m\) being \(\operatorname{Max}_{x \in J} t_j\). It is easily shown that \(\omega V_i = \bigcup_{\pi^{-1}(i)} V_j\) \((i \in \pi J)\), while \(\omega V_i \subseteq C'\) \((i \in \pi J)\). The consequence is \((f \times \omega) L^n c L^n (n \in \mathbb{Z})\), with an induced homomorphism \(h_p(L^n, L^m) \to h_p(L'^n, L'^m)\) \((m \geq n)\). The map \((-1)^{\text{number of members in } \pi J}\).

\(L^n, n+1\) induces the required \(E^n_{r,j} \to E^n_{r,j} (n, j \in \mathbb{Z}; r = 1, 2, \cdots)\). (The power of \((-1)\) is the degree of \(\omega\).)

The homology sheaf. Let \(\mathcal{P} = \{X, A \cup (X - U_i)\,|\,i \in I\} = \mathcal{P}_U\) for some finite open cover \(\mathcal{U} = \{U_i\,|\,i \in I\}\) of \(X\), \(A\) being some subspace. Evidently

\[
h_\ast \left( \bigcap X_i, \bigcap A_i \right) = h_\ast(X, A),
\]

\[
C^\ast(\mathcal{P}_U; h_\ast(\bigcup \cdot)) = C^\ast(\mathcal{U}; h^X_\ast A),
\]

where \(h^X_\ast A\) is the graded presheaf \(\{h_\ast(X, A \cup (X - U))\,|\,\text{open } U \subset X\}\). Thus, we obtain a spectral sequence

\[
E^n_{2,r} \cong H^n(\mathcal{U}; h^X_n A) \Rightarrow h_{n-r}(X, A).
\]
For $X$ compact, the direct limit of $(f)$, as $\mathcal{U}$ is refined, is a spectral sequence

$E^n_{2i,j} \cong H^n(X; \mathcal{H}_j^{X,A}) \Rightarrow h_{j-n}(X, A),\]

where $\mathcal{H}_j^{X,A}$ is the induced sheaf of $h_j^{X,A}$. As $A$ approximates an open set $V$ from within, the direct limit of $(g)$ is

$h^X(*) \cong \mathcal{H}_0(X, V; \mathcal{H}_j^X) \Rightarrow h_{j-n}(X, V)\

$E^n_{2i,j} \cong H^n(X, V; \mathcal{H}_j^X) \Rightarrow h_{j-n}(X, V).$

$h^X_*$ is called the homology sheaf of $X$. If $h_j^X \cong \{0\}$ except for $j=j_0$ ($= \text{some integer}$), e.g., if $X$ is a $j_0$-manifold and $h_*$ is standard, then $(h)$ collapses to a family of isomorphisms

$H^n(X, V; \mathcal{H}_j^{X}) \cong h_{j_0-n}(X, V) \quad (n \in \mathbb{Z}).$

(Compare to [2].)

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