CLASS NUMBERS AND $\mu$-INVARIANTS OF CYCLOTOMIC FIELDS

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Abstract. We give a new upper bound for the $\mu$-invariant of a cyclotomic field by estimating the first factor of the class number of the $\mathfrak{p}$th cyclotomic field ($\mathfrak{p}$ an odd prime).

For each $n \geq 0$ let $h_n$ denote the class number of the cyclotomic field of $p^{n+1}$th roots of unity, where $p$ is an odd prime. According to Iwasawa [1], the greatest exponent $e(n)$ for which $p^{e(n)} | h_n$ is given by a formula

$$e(n) = \lambda n + \mu p^n + v,$$

valid for all sufficiently large $n$. Here $\lambda$, $\mu$, and $v$ are integers ($\lambda, \mu \geq 0$) independent of $n$. In [2] Iwasawa proved the following estimates for $\mu$:

(i) $\mu < p - 1$ for all $p$,

(ii) if $c > \frac{1}{3}$, then there exists a bound $N(c)$ such that $\mu < c(p-1)$ whenever $p > N(c)$.

We shall show that $\mu < (p-1)/2$ for all $p$.

Let us denote by $h^ -$ the so-called first factor of $h_0$. As shown in [2], the problem of estimating $\mu$ can be reduced to that of estimating $h^ -$ by means of the relation $p^{\mu/2} \leq h^-$. It is known that

$$h^- = (2p)^{-p-3/2} \left| \prod_{\chi \in S} \sum_{n=1}^{p-1} \chi(n)n \right|,$$

where $S$ denotes the set of all odd residue class characters mod $p$. Noting that

$$\sum_{\chi \in S} \chi(m)\chi'(n) = \begin{cases} (p - 1)/2 & \text{if } m \equiv n \text{ (mod } p), (mn, p) = 1, \\ -(p - 1)/2 & \text{if } m \equiv -n \text{ (mod } p), (mn, p) = 1, \\ 0 & \text{otherwise}, \end{cases}$$

Received by the editors August 17, 1973.


Key words and phrases. Class number, cyclotomic field.

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$(\chi'$ means the complex conjugate of $\chi$), we first get
\[
\sum_{\chi \in S} \left| \sum_{n=1}^{p-1} \chi(n)n \right|^2 = ((p - 1)/2) \left( \sum_{n=1}^{p-1} n^2 - \sum_{n=1}^{p-1} n(p - n) \right) = (p - 2)(p - 1)^2 p/12.
\]

Therefore, by the arithmetic-geometric mean inequality,
\[
\prod_{\chi \in S} \left| \sum_{n=1}^{p-1} \chi(n)n \right|^{4/(p-1)} \leq (p - 2)(p - 1)p/6 < p^3/6.
\]

This gives us the estimate

\[ h^- < 2p(p/24)^{(p-1)/4} \]

Thus, if $p>3$, we see that $h^- < p^{(p-1)/4}$ and so $\mu < (p-1)/2$. This holds also for $p=3$, since then $h^- = 1$. (As a matter of fact, we know that $\mu = 0$ for all regular primes.)

It should be mentioned that the result (1) has been obtained earlier by Lepistö [3] and the author [4] by more complicated methods than that presented above.

**References**


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