ON THE DOMINATED ERGODIC THEOREM
IN $L_2$ SPACE

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Abstract. Let $T$ be a contraction on $L_2$ of a $\sigma$-finite measure space, $A_n(T)$ the operator $(1/n)(T^n + \cdots + T)$, $S(T)f$ the function $\sup_{n>0}|A_n(T)f|$. Theorem 1. Assume that, whatever be the measure space, $S(U)f \in L_2$ for each unitary operator $U$ on $L_2$ and each function $f \in L_2$. Then there exists a universal constant $K$ such that $\|S(T)f\| \leq K \|f\|$ for each contraction $T$ on $L_2$ and each $f \in L_2$. Theorem 2. Let $T$ be a contraction on $L_2$ and let $U$ be a unitary dilation of $T$ acting on a Hilbert space $H$ containing $L_2$. If all expressions of the form $\sum_{n=1}^{\infty} P_n A_n(U)$, where $P_n$ are mutually orthogonal projections, are bounded operators on $H$, then for each $f \in L_2$, $S(T)f \in L_2$ and $A_n(T)f$ converges a.e.

Let $T$ be a contraction on $L_2$ of a $\sigma$-finite measure space $(X, \mathcal{F}, \mu)$. $A_n(T)$ is the operator $(1/n)(T^n + \cdots + T^n)$. If $f$ is a function in $L_2$, let $S(T)f$ be the function $\sup_{n>0}|A_n(T)f|$. One of the unresolved problems of ergodic theory is whether $A_n(T)f$ converges almost everywhere for each $f$ in $L_2$. It is known that the result would be implied by the dominated ergodic theorem; i.e., the existence of a constant $K$ such that for all $f \in L_2$

\[(1) \quad \|S(T)f\| \leq K \|f\|.
\]

We show below that if the dominated ergodic theorem holds for unitary operators (\equiv invertible isometries) on $L_2$, then it holds for contractions. Whether the theorem holds for unitary operators on $L_2$ is still not known. It may be pointed out that the dominated ergodic theorem holds for positive unitary operators (but perhaps not for positive contractions) on $L_2$, as shown by E. M. Stein (see [4, p. 367] and [5, p. 87]), and for invertible, not necessarily positive, isometries on $L_p$, $1 < p < \infty$, $p \neq 2$, as shown by Mrs. A. Ionescu Tulcea [4]. (Positive means $f \geq 0$ implies

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For positive contractions on $L_2$, partial results were obtained by R. V. Chacon and J. Olsen [2], and Chacon and S. M. McGrath [1].

We also prove a result about individual contractions which may be of interest should the general conjecture prove to be false. Given an operator $U$, call $V$ a decomposition of $U$ iff $V$ is of the form $\sum_{n=1}^{\infty} P_n A_n(U)$, where $P_n$ are mutually orthogonal projections. Let $T$ be a contraction on $L_2$ and $U$ a unitary dilation of $T$ acting on a space $H \supseteq L_2$. We prove that if all decompositions of $U$ are bounded operators on $H$ then $S(T)f \in L_2$ for each $f \in L_2$, and hence $A_n(T)f$ converges almost everywhere.

**Theorem 1.** Assume that, whatever be the space $L_2$, $S(U)f \in L_2$ for each unitary operator $U$ on $L_2$ and each function $f \in L_2$. Then there exists a constant $K$ such that (1) holds for each contraction $T$ on $L_2$ and each $f \in L_2$.

**Proof.** We first observe that there is a constant $K$ such that

$$\|S(U)f\| \leq K \|f\| \quad (2)$$

for all $f \in L_2$ and all unitary operators $U$ on $L_2$. Otherwise for $n = 1, 2, \ldots$, there would exist measure spaces $(X_n, \mathcal{F}_n, \mu_n)$, unitary operators $U_n$ on $L_2(X_n, \mathcal{F}_n, \mu_n)$, and functions $g_n \in L_2(X_n, \mathcal{F}_n, \mu_n)$, such that $\|g_n\| = 1/n$ and $\|S(U_n)g_n\| \geq 1$. Let $(X, \mathcal{F}, \mu)$ be the direct sum $\bigoplus_{n=1}^{\infty} (X_n, \mathcal{F}_n, \mu_n)$, and represent a function $f$ on $X$ as $f = (f_n)$, where $f_n$ is the restriction of $f$ to $X_n$. Define a unitary operator $U$ on $L_2(X, \mathcal{F}, \mu)$ by $U(f_n) = (U_n f_n)$. Then

$$g = (g_n) \in L_2(X, \mathcal{F}, \mu),$$

but

$$\|S(U)g\|^2 = \left\|\sum_{n=1}^{\infty} S(U_n)g_n\right\|^2 = \infty,$$

which is a contradiction. Thus (2) holds. Now let $E_1, E_2, \ldots$, be disjoint measurable sets; write $1_E$ for the indicator function of a set $E$. For each unitary operator $U$, each $f \in L_2$

$$\left|\sum_{n=1}^{\infty} 1_{E_n} A_n(U)f\right| \leq S(U)f \quad \text{a.e.}, \quad (3)$$

which implies

$$\left\|\sum_{n=1}^{\infty} 1_{E_n} A_n(U)\right\| \leq K, \quad (4)$$

where in (4) $1_{E_n}$ are projection operators corresponding to the multiplication by $1_{E_n}$, and $K$ is the constant appearing in (2). Next observe that (4) may be generalized to

$$\left\|\sum_{n=1}^{\infty} P_n A_n(U)\right\| \leq K, \quad (5)$$
where $U$ is a unitary operator on an arbitrary Hilbert space $H$, $(P_n)$ is any sequence of mutually orthogonal projections on $H$, and $K$ is again the constant in (2). This follows from the fact that given $H$ and $(P_n)$, there exists $L_2(X, \mathcal{F}, \mu)$ isometrically isomorphic to $H$ and such that under the isomorphism, $P_n$ becomes $1_{E_n}$. Now let $T$ be any contraction on $L_2$ of an arbitrary measure space $(X, \mathcal{F}, \mu)$. Then

$$\left\| \sum_{n=1}^{\infty} 1_{E_n} A_n(T) \right\| \leq K$$

for any sequence of disjoint sets $E_n$ in $\mathcal{F}$, where $K$ is as before. To prove (6), apply the dilation theorem of Sz.-Nagy (cf. [6] or [3]; dilations in [6] are power dilations in the terminology of [3]): there exists a Hilbert space $H$, a projection $P$, and a unitary operator $U$ on $H$ such that $PH=L_2(X, \mathcal{F}, \mu)$ and $T^n=PU^n$ for $n=1, 2, \cdots$. Then $A_n(T)=PA_n(U)$, hence $1_{E_n} A_n(T)=1_{E_n} PA_n(U)$. $P_n=1_{E_n} P$ form a set of mutually orthogonal projections, and therefore (6) follows from (5). Finally, to conclude the proof of the theorem, note that given $T$ and $f$ there exist mutually disjoint sets $E_1, E_2, \cdots$ such that $S(T)f=|\sum_{n=1}^{\infty} 1_{E_n} A_n(T)f|$. Hence

$$\|S(T)f\| \leq \left\| \sum_{n=1}^{\infty} 1_{E_n} A_n(T) \right\| \cdot \|P\| \cdot \|f\| \leq K \|f\|.$$

Now consider the case of an individual contraction $T$.

**Theorem 2.** Let $T$ be a contraction on $L_2(X, \mathcal{F}, \mu)$ and let $U$ be a unitary dilation of $T$ acting on a Hilbert space $H$. If all decompositions of $U$ are bounded operators on $H$, then, for each $f \in L_2$, $S(T)f \in L_2$ and $A_n(T)f$ converges a.e.

**Proof.** Let $f \in L_2$, $E_n=\{x \in X: |A_n(T)f|=S(T)f\}$, $P_n=1_{E_n} P$, where $P$ is a projection such that $T^n=PU^n$, $n=1, 2, \cdots$. Disjoint the sets $E_n$ if necessary. If there exists a constant $C=C(T, f)$ such that

$$\left\| \sum_{n=1}^{\infty} P_n A_n(U) \right\| \leq C,$$

then the relation (7) holds with $C$ replacing $K$, showing that $S(T)f \in L_2$. The proof of the pointwise convergence of $A_n(f)$ is like in [1] or [4].

**Remark.** It is clearly of interest to consider only the minimal unitary dilation $U_0$ of $T$, since each unitary dilation reduces to $U_0$ on $\bigvee_{n=0}^{\infty} U_0^n L_2$ (cf. [6]).

We finally note that sequences of successive Cesàro averages of powers of operators in Theorems 1 and 2 may be replaced by sequences of any linear combinations of powers; the proofs remain the same.
Added in proof. Professor D. L. Burkholder has pointed out to us that a result of his, a consequence of his theory of semi-Gaussian spaces (Theorem 2, p. 128, Trans. Amer. Math. Soc. 104 (1962), implies that the dominated ergodic theorem fails for contractions on $L_2$. Combining this with the results of the present paper, one obtains that the answer to the question about unitary operators raised in the introduction is negative: There exists a unitary operator on $L_2$ for which the dominated ergodic theorem fails.

REFERENCES


