ON THE WEIGHT OF A TOPOLOGICAL SPACE

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Abstract. The main result in this paper states that the weight of a regular space is the product of three cardinal functions, namely the Lindelöf degree, the pluming degree, and the point separating weight.

1. Introduction. In recent years there has been considerable interest in cardinal functions, no doubt due in large part to Arhangel'skii's recent solution [3] of Alexandroff's problem. (See Comfort's paper [6] for a survey of cardinal invariants; the fundamental tract on cardinal functions is Juhasz [9].) The main result of this paper states that the weight of a regular space is the product of three cardinal functions. This result is motivated by the following three theorems, due respectively to Ponomarev [13], Arhangel'skii [4], and Nagata [12]. (1) Every Lindelöf p-space with a point-countable base has a countable base. (2) If X is a p-space and \( \mathcal{N} \) is a net for X, then there is a base \( \mathcal{B} \) for X with \( |\mathcal{B}| \leq |\mathcal{N}| \). (3) Every paracompact p-space with a point-countable separating open cover is metrizable. We also study conditions under which the weight of an infinite Hausdorff space X is not greater than \( |X| \).

Throughout this paper m and n denote cardinal numbers, \( \sigma \) denotes an ordinal number, and \( |A| \) denotes the cardinality of a set A. All regular spaces are T1 and all p-spaces are T1 and completely regular.

2. The main theorem. We begin with some definitions and known results which are needed to state and prove the main theorem. Let X be a set, let \( \mathcal{S} \) be a cover of X. The cover \( \mathcal{S} \) is said to be separating if given distinct points \( p \) and \( q \) in X, there is some \( G \) in \( \mathcal{S} \) such that \( p \notin G, q \notin G \). For \( p \) in X, the order of \( p \) with respect to \( \mathcal{S} \), denoted ord(\( p, \mathcal{S} \)), is the cardinality of the set \( \{ G \in \mathcal{S} : p \in G \} \).

Now let X be a topological space. A collection \( \mathcal{N} \) of subsets of X is a net for X if for each point \( p \) in X and each neighborhood \( R \) of \( p \), there is some \( N \) in \( \mathcal{N} \) such that \( p \in N \subseteq R \). The weight of X, denoted \( w(X) \), is \( \aleph_0 \cdot m \), where \( m \) is the smallest cardinal number which arises as the cardinality of some base for X. The Lindelöf degree of X, denoted \( L(X) \),
is $\aleph_0 \cdot m$, where $m$ is the smallest cardinal such that every open cover of $X$ has a subcover of cardinality $\leq m$.

Now suppose $X$ is a $T_1$ space. Let $m$ be the smallest cardinal such that $X$ has a separating open cover $\mathscr{S}$ with $\text{ord}(p, \mathscr{S}) \leq m$ for all $p$ in $X$. (Since $X$ is $T_1$, it is easy to see that $m$ exists and $m \leq |X|$.) The point separating weight of $X$, denoted $\text{psw}(X)$, is $\aleph_0 \cdot m$.

We now extend the concept of a $p$-space to higher cardinality by introducing the pluming degree of a regular space. Our definition is based on an internal characterization of $p$-spaces due to Burke [5]. A collection $\{\mathcal{G}_\alpha : \alpha \in A\}$ of open covers of a topological space $X$ is a pluming for $X$ if the following holds: if $p \in G_\alpha \in \mathcal{G}_\alpha$ for all $\alpha$ in $A$, then (a) $C_p = \bigcap_{\alpha \in A} G_\alpha$ is compact; (b) $\{\bigcap_{\beta \in F} G_\beta : F$ a finite subset of $A\}$ is a “base” for $C_p$ in the sense that given any open set $R$ containing $C_p$, there is a finite subset $F$ of $A$ with $\bigcap_{\beta \in F} G_\beta \subseteq R$. Before defining the pluming degree we need the following existence result.

**Proposition 2.1.** Let $X$ be a regular space. Then $X$ has a pluming $\{\mathcal{G}_\alpha : \alpha \in A\}$ with $|A| \leq w(X)$.

**Proof.** Let $\mathcal{B} = \{B_\alpha : \alpha \in A\}$ be a base for $X$ with $|A| \leq w(X)$. Let $D = \{(\alpha, \beta) \in A \times A : B_\alpha \subseteq B_\beta\}$, and for each $(\alpha, \beta)$ in $D$ let $\mathcal{G}(\alpha, \beta) = \{X - B_\alpha, B_\beta\}$. Then $\{\mathcal{G}(\alpha, \beta) : (\alpha, \beta) \in D\}$ is a pluming for $X$ with $|D| \leq w(X)$. □

For a regular space $X$, the pluming degree of $X$, denoted $p(X)$, is $\aleph_0 \cdot m$, where $m$ is the smallest cardinal such that $X$ has a pluming $\{\mathcal{G}_\alpha : \alpha \in A\}$ with $|A| = m$. By the above proposition, $p(X) \leq w(X)$. Also, it follows from Burke’s theorem [5] that for a completely regular space $X$, $p(X) = \aleph_0$ if and only if $X$ is a $p$-space.

The following two lemmas are used in the proof of the main theorem. (See [11], [7], and [8].)

**Lemma 2.2 (Miščenko).** Let $X$ be a set, let $m$ be an infinite cardinal, let $\mathcal{S}$ be a collection of subsets of $X$ such that $\text{ord}(p, \mathcal{S}) \leq m$ for all $p$ in $X$, and let $H$ be a subset of $X$. Then the cardinality of the set of all finite minimal covers of $H$ by elements of $\mathcal{S}$ does not exceed $m$.

**Lemma 2.3 (Holsztyński).** Let $X$ be a Hausdorff space, let $\mathcal{N}$ be a net for $X$ with $|\mathcal{N}| \leq m$. Then there is an open cover $\mathcal{U}$ of $X$ with $|\mathcal{U}| \leq m$ such that given distinct points $p$ and $q$ in $X$, there exists $U$ in $\mathcal{U}$ such that $p \in U$, $q \notin U$.

**Theorem 2.4.** Let $X$ be a regular space. Then

$$w(X) = L(X) \cdot p(X) \cdot \text{psw}(X).$$
Proof. Clearly $L(X) \cdot p(X) \cdot \text{psw}(X) \leq w(X)$. Suppose, then, that $L(X) \cdot p(X) \cdot \text{psw}(X) = m$, and let us construct a base $\mathcal{B}$ for $X$ with $|\mathcal{B}| \leq m$. Let $\{G_\alpha : \alpha \in A\}$ be a pluming for $X$ with $|A| \leq m$. Since $L(X) \leq m$, we may assume that $|G_\alpha| \leq m$ for each $\alpha$ in $A$. Let $\mathcal{G} = \bigcup_{\alpha \in A} G_\alpha$ and let $\mathcal{H}$ be all finite intersections of elements of $\mathcal{G}$. Note that $|\mathcal{H}| \leq m$. Let $\mathcal{I}$ be a separating open cover of $X$ such that $\text{ord}(\mathcal{I}, \mathcal{G}) \leq m$ for all $p$ in $X$. We may assume that $X \in \mathcal{P}$, and hence for any subset $H$ of $X$ there is at least one finite minimal cover of $H$ by elements of $\mathcal{I}$, namely $\{X\}$.

First we shall prove that $|\mathcal{I}| \leq m$. For each $H$ in $\mathcal{H}$ let $\{\mathcal{I}(H, \sigma): 0 \leq \sigma < n_H \leq m\}$ be all finite minimal covers of $H$ by elements of $\mathcal{I}$ (use Miščenko’s lemma), and let $\mathcal{I}' = \bigcup \{\mathcal{I}(H, \sigma): H \in \mathcal{H}, 0 \leq \sigma < n_H\}$. We shall show that $\mathcal{I} \subseteq \mathcal{I}'$, from which it follows that $|\mathcal{I}| \leq m$. Let $S_0 \in \mathcal{I}$, and let $p \in S_0$. For each $\alpha$ in $A$ choose $G_\alpha$ in $\mathcal{G}$ such that $p \in G_\alpha$, and let $C_p = \bigcap_{\alpha \in A} G_\alpha$. Recall that $C_p$ is compact. Construct a subcollection $\mathcal{I}_0$ of $\mathcal{I}$ which covers $C_p$ as follows: $S_0 \in \mathcal{I}_0$, and for each $q$ in $(C_p - S_0)$ choose an element of $\mathcal{G}$ which contains $q$ but not $p$. Let $\mathcal{I}_1$ be a finite subcollection of $\mathcal{I}_0$ which covers $C_p$. Let $F$ be a finite subset of $A$ such that $\bigcap_{\alpha \in F} G_\alpha \subseteq \bigcup \mathcal{I}_1$, and let $H = \bigcap_{\alpha \in F} G_\alpha$. Let $\mathcal{I}_2$ be a minimal subcollection of $\mathcal{I}_1$ which covers $H$, and note that $S_0 \in \mathcal{I}_2$. Now $\mathcal{I}_2 = \mathcal{I}(H, \sigma)$ for some $\sigma < n_H$, and so $S_0 \in \mathcal{I}'$.

Next let us construct a net $\mathcal{N}$ for $X$ with $|\mathcal{N}| \leq m$. Let $\mathcal{N} = \{H - W: H \in \mathcal{H}, W = \emptyset$ or $W$ a finite union of elements of $\mathcal{I}\}$. Clearly $|\mathcal{N}| \leq m$. To see that $\mathcal{N}$ is a net, let $p \in X$, let $R$ be an open neighborhood of $p$. For each $\alpha$ in $A$ choose $G_\alpha$ in $\mathcal{G}$ such that $p \in G_\alpha$, and let $C_p = \bigcap_{\alpha \in A} G_\alpha$. First suppose $C_p \subseteq R$. Choose a finite subset $F$ of $A$ such that $\bigcap_{\alpha \in F} G_\alpha \subseteq R$. Then $H = \bigcap_{\alpha \in F} G_\alpha$ is an element of $\mathcal{N}$ and $p \in H \subseteq R$. Next, assume that $C_p \cap R = \emptyset$. Let $\mathcal{I}_0$ be a finite subcollection of $\mathcal{I}$ which covers $C_p$ such that $p \notin \bigcup \mathcal{I}_0 = W$. Now $C_p \subseteq R \cup W$, so there is a finite subset $F$ of $A$ such that $\bigcap_{\alpha \in F} G_\alpha \subseteq W \cup R$. Let $H = \bigcap_{\alpha \in F} G_\alpha$. Then $H - W$ is an element of $\mathcal{N}$ such that $p \in H - W \subseteq R$.

Finally, we construct a base $\mathcal{B}$ for $X$ with $|\mathcal{B}| \leq m$. Since $X$ has a net $\mathcal{N}$ with $|\mathcal{N}| \leq m$, it follows from Lemma 2.3 that there is an open cover $\mathcal{U}$ of $X$ with $|\mathcal{U}| \leq m$ such that given distinct points $p, q$ in $X$, there exists $U$ in $\mathcal{U}$ such that $p \in U$, $q \notin U$. We may assume that $\mathcal{U}$ is closed under finite intersections and $X \in \mathcal{U}$. Let $\mathcal{B} = \{H \cap U: H \in \mathcal{H}, U \in \mathcal{U}\}$. It is clear that $\mathcal{B}$ is an open collection with $|\mathcal{B}| \leq m$, and so it remains to show $\mathcal{B}$ a base. Let $p \in X$, let $R$ be an open neighborhood of $p$. For each $\alpha$ in $A$ choose $G_\alpha$ in $\mathcal{G}$ such that $p \in G_\alpha$, and let $C_p = \bigcap_{\alpha \in A} G_\alpha$. If $C_p \subseteq R$, the result is clear. Suppose, then, that $C_p \cap R = \emptyset$. Let $U$ and $W$ be disjoint open sets such that $p \in U \in \mathcal{U}$ and $C_p \cap R \subseteq W$. Now $C_p \subseteq W \cup R$, so
there is a finite subset \( F \) of \( A \) with \( \bigcap_{x \in F} G_x \subseteq W \cup R \). Hence

\[
B = \left( \bigcap_{x \in F} G_x \right) \cap U
\]

is an element of \( \mathcal{B} \) such that \( p \in B \subseteq R \). This completes the proof.

**Corollary 2.5.** Every Lindelöf \( p \)-space with a point-countable separating open cover has a countable base.

**Corollary 2.6.** Let \( m \) be infinite, let \( X \) be a regular space. Suppose \( X = \bigcup_{x \in A} X_x \), where \( |A| \leq m \) and \( w(X_x) \leq m \) for all \( x \) in \( A \). Then \( w(X) \leq m \) if and only if \( p(X) \leq m \).

**Proof.** Suppose \( p(X) \leq m \). Clearly \( X \) has a net \( \mathcal{N} \) with \( |\mathcal{N}| \leq m \). It easily follows that \( L(X) \leq m \), and by Lemma 2.3 \( X \) has a separating open cover \( \mathcal{U} \) with \( |\mathcal{U}| \leq m \). Hence \( w(X) \leq m \).

**Remark 2.7.** Corollary 2.5 gives a generalization of the result by Ponomarev mentioned in §1. Corollary 2.6 is a slight extension of the beautiful Weight Addition Theorem of Arhangel’skii [4].

3. **Conditions under which** \( w(X) \leq |X| \). It is well known that \( w(X) \leq 2^{|X|} \) for any infinite topological space \( X \). However, as we now show, \( w(X) \leq |X| \) under fairly weak assumptions on \( X \). The following result is basic.

**Theorem 3.1.** Let \( X \) be an infinite Hausdorff space. Then \( w(X) \leq |X| \) if and only if there is a cover \( \mathcal{H} \) of \( X \) such that each \( K \) in \( \mathcal{H} \) is compact and has a base consisting of not more than \( |X| \) open sets.

**Proof.** If \( w(X) \leq |X| \), the existence of \( \mathcal{H} \) is obvious. Assume, then, that \( \mathcal{H} \) is a cover of \( X \) having the stated properties. Clearly we may assume \( |\mathcal{H}| \leq |X| \). For each \( K \) in \( \mathcal{H} \) let \( \{ V(K, \sigma) : 0 \leq \sigma < m_K \leq |X| \} \) be a base for \( K \). Now \( X \) has a net \( \mathcal{N} \) with \( |\mathcal{N}| \leq |X| \), so by Lemma 2.3 there is an open cover \( \mathcal{U} \) of \( X \) with \( |\mathcal{U}| \leq |X| \) such that given distinct points \( p, q \) in \( X \), there exists \( U \) in \( \mathcal{U} \) such that \( p \in U \), \( q \notin U \). We may assume that \( \mathcal{U} \) is closed under finite intersections and \( X \) in \( \mathcal{U} \). Let \( \mathcal{B} = \{ U \cap V(K, \sigma) : U \in \mathcal{U}, K \in \mathcal{H}, 0 \leq \sigma < m_K \} \). Then \( \mathcal{B} \) is a base for \( X \) with \( |\mathcal{B}| \leq |X| \).

**Remark 3.2.** Let \( X \) be a topological space. Recall that \( X \) is of point-countable type [2] if each point of \( X \) belongs to a compact set having a countable base, and \( X \) is a \( q \)-space [10] if each point \( p \) of \( X \) has a countable collection \( \{ U(n, p) : n = 1, 2, \ldots \} \) of open neighborhoods such that if \( x_n \in U(n, p) \) for \( n = 1, 2, \ldots \), then \( \langle x_n \rangle \) has a cluster point. It is clear from Theorem 3.1 that for every infinite Hausdorff space \( X \) of point-countable type, \( w(X) \leq |X| \). (This result has recently been announced by Nagami.) It is not difficult to show that every regular \( q \)-space in which every countably
compact space is compact is a space of point-countable type. We thus have the following corollary of Theorem 3.1. (See [14] for the definition of \( \theta \)-refinable; a topological space \( X \) is meta-Lindelöf if every open cover has a point-countable open refinement. See [1].)

**Corollary 3.3.** Let \( X \) be an infinite regular \( q \)-space which is either \( \theta \)-refinable or meta-Lindelöf. Then \( w(X) \leq |X| \).

**Problem 3.4.** Let \( m \) be a cardinal, \( m > \aleph_0 \). Is there a countably compact regular space \( X \) such that \( |X| = m, w(X) > m \)? In a private communication to the author, W. W. Comfort constructed, for each cardinal \( m \) such that \( m^{\aleph_0} = m \), a countably compact completely regular space \( X \) with \( |X| = m \) and \( w(X) > m \).

**REFERENCES**