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# LUSIN AND SUSLIN TOPOLOGIES ON A SET

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ABSTRACT. It is proved that the Borel subsets of two Suslin topologies on a set $X$ are identical if and only if there is a Suslin topology on $X$ finer than both the given topologies. Properties of the supremum topology are given under suitable conditions on the subdivisions of the spaces.

A Hausdorff topological space is said to be Lusin (resp. Suslin) if there is a continuous bijective mapping (resp. a continuous surjective mapping) from a complete separable metric space onto $X$. The topological spaces which are Suslin or Lusin have a number of interesting measure theoretic properties. We mention two of them here. The first and perhaps one of the very interesting properties is that if $X$ is a Lusin (resp. Suslin) space and $X$ is a subspace of a Hausdorff topological space $Y$, then $X$ is Borel in $Y$ (resp. $X$ is measurable for every finite Borel measure on $Y$). The second property is that every finite Borel measure on a Suslin space is a Radon measure. The development of the Suslin spaces and, in particular, the proofs of the above results can be found in [1].

In this note we consider a set $X$ with two Suslin topologies $\tau_1$ and $\tau_2$. It is of importance and interest to know when the $\tau_1$-Borel sets and $\tau_2$-Borel sets are identical. It is known that this happens if the $\inf(\tau_1, \tau_2)$ is a Hausdorff topology. We shall show that a necessary and sufficient condition for the above to happen is that $\sup(\tau_1, \tau_2)$ is a Suslin topology (Theorem 1). We also prove some properties of the supremum topology in terms of conditions on subdivisions on the space.

We first have the

THEOREM 1. *Let $\tau_1$ and $\tau_2$ be two Suslin (resp. Lusin) topologies on a set $X$. The $\tau_1$-Borel sets and $\tau_2$-Borel sets are identical if and only if the* $\sup(\tau_1, \tau_2)$ *is a Suslin (resp. Lusin) topology on $X$.*

PROOF. Assume that $\sup(\tau_1, \tau_2)$ is a Suslin topology on $X$. It is well known that the Borel sets of two comparable Suslin topologies on a set

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are identical [1]. We deduce that $\sup(\tau_1, \tau_2)$, $\tau_1$ and $\tau_2$ all have the same Borel sets. Conversely, assume $\tau_1$-Borel sets and $\tau_2$-Borel sets are identical. Let $\Delta$ be a diagonal in $X\times X$. Then, $\Delta$ is closed in the product space $(X\times X, \tau_1\times\tau_1)$. Also the complement of $\Delta$ which is $\tau_1\times\tau_1$ open is Lindelöf and hence can be covered by a countable number of $\tau_1\times\tau_1$ open rectangles of the form $V\times W$. However, $W$ is $\tau_2$-Borel and each such set $V\times W$ is $\tau_1\times\tau_2$-Borel. It follows that $(X\times X)\backslash\Delta$ is $\tau_1\times\tau_2$-Borel and hence $\Delta$ is $\tau_1\times\tau_2$-Borel. But since $(X\times X, \tau_1\times\tau_2)$ is Suslin (resp. Lusin), we deduce that $\Delta$ with the induced $\tau_1\times\tau_2$-topology is Suslin (resp. Lusin). Also, $X$ with the $\sup(\tau_1, \tau_2)$-topology is homeomorphic to $\Delta$ with the induced $\tau_1\times\tau_2$-topology. The proof is complete.

COROLLARY *For $X$, $\tau_1$, $\tau_2$ as above, if* $\inf(\tau_1, \tau_2)$ *is Hausdorff the n* $\sup(\tau_1, \tau_2)$ *is a Suslin topology on $X$.*

The Corollary is deduced as an immediate consequence by observing that under the additional hypothesis $\tau_1$-Borel sets and $\tau_2$-Borel sets are identical.

We now recall

DEFINITION 1. Let $(X, \tau)$ be a Hausdorff topological space. A *subdivision* $\{(\mathscr{C}_n, p_n, \varphi_n)\}$ on $X$ is the following system.

(a) $\mathscr{C}_n$, for every $n=0$ to $\infty$, is a countable set and $p_n$ is a surjection from $\mathscr{C}_{n+1}$ to $\mathscr{C}_n$.

(b) For every $n=0$ to $\infty$, $\varphi_n$ is a one-one mapping of $\mathscr{C}_n$ into the collection of nonvoid subsets of $X$ such that $\bigcup\{\varphi_0(c): c\in\mathscr{C}_0\}=X$ and for every $n\in N$ and every $c\in\mathscr{C}_n$, $\bigcup\{\varphi_{n+1}(d): d\in\mathscr{C}_{n+1}, p_n(d)=c\}=\varphi_n(c)$.

(c) For every coherent sequence $\{c_n\}$ viz. $c_n\in\mathscr{C}_n$ such that $p_n(c_n)=c_{n-1}$, by (b), $\{\varphi_n(c_n)\}$ forms a base of a filter and this filter converges to an element $x$ in $X$ such that $x$ belongs to $\varphi_n(c_n)$ for each $n$.

(d) Every element $x$ in $X$ is the limit of at least one filter as in (c).

Further the subdivision $\{(\mathscr{C}_n, p_n, \varphi_n)\}$ is said to be a *strict subdivision* if for every $n$, $\{\varphi_n(c)\}_{c\in\mathscr{C}_n}$ is a sequence of mutually disjoint subsets of $X$.

The following theorem is a well-known result about Suslin and Lusin spaces [1].

THEOREM S. *A Hausdorff topological space $(X, \tau)$ is Suslin (resp. Lusin) if and only if $(X, \tau)$ admits a subdivision (resp. a strict subdivision).*

DEFINITION 2. Let $(X, \tau)$ be a topological space and $\{(\mathscr{C}_n, p_n, \varphi_n)\}$ a subdivision on $X$. The topology on $X$ with the subbase for open sets consisting of $\{\varphi_n(c): c\in\mathscr{C}_n, n\in N\}$ is called the topology associated with the subdivision.

LEMMA 1. *Let $(X, \tau)$ be a Lusin space with a strict subdivision $\{(\mathscr{C}_n, p_n, \varphi_n)\}$. Then the topology $\tau'$ associated with this subdivision is the finest among the topologies on $X$ for which $\{(\mathscr{C}_n, p_n, \varphi_n)\}$ is a subdivision. Consequently $(X, \tau')$ is Lusin and $\tau$-Borel sets and $\tau'$-Borel sets are identical.*

PROOF. Let $\alpha$ be any Hausdorff topology on $X$ such that $\{(\mathscr{C}_n, p_n, \varphi_n)\}$ is a subdivision for $(X, \alpha)$. Then, by the condition (d) of the definition of subdivision, given $x$ in $X$ and an $\alpha$-neighborhood $V$ of $x$, there is a $c \in \mathscr{C}_n$ such that $x$ is in $\varphi_n(c)$ and $\varphi_n(c) \subset V$. Hence, the topology $\tau'$ is finer than $\alpha$ on $X$. Since there is at least one such topology viz. $\tau$, we conclude that $(X, \tau')$ is Hausdorff.

Now since the subdivision $\{(\mathscr{C}_n, p_n, \varphi_n)\}$ is strict, we observe that $\{\varphi_n(c) : c \in \mathscr{C}_n, n \in N\}$ is in fact, a base for the topology $\tau'$. Also, for any $x$ in $X$ there is exactly one coherent sequence $\{c_n\}$ such that $\varphi_n(c_n)$ contains $x$ for every $n$. Clearly, $\{\varphi_n(c_n)\}$ for this coherent sequence is precisely a base of $\tau'$-neighborhoods of $x$ and we conclude that condition (d) of the definition of subdivision is satisfied by $\{(\mathscr{C}_n, p_n, \varphi_n)\}$ relative to $(X, \tau')$. On the other hand, given any coherent sequence $\{c_r\}$ there is a unique $x$ in $X$ which is contained in all $\varphi_n(c_n)$ and to which $\{\varphi_n(c_n)\}$ converges in the $\tau$-topology. By the definition, any $\tau'$-neighborhood of $x$ necessarily contains $\varphi_m(c_m)$ for some $m$ and we conclude that $\{\varphi_n(c_n)\}$ converges to $x$ in $\tau'$. Hence $\{(\mathscr{C}_n, p_n, \varphi_n)\}$ is a strict subdivision for $(X, \tau')$. The proof of the Lemma is now completed easily.

THEOREM 2. *Let $\tau_1$ and $\tau_2$ be two topologies on a set $X$ such that $(X, \tau_1)$ and $(X, \tau_2)$ are Lusin spaces. Then the following are equivalent:*

(2.1) *There is a topology $\tau$ on $X$ finer than both $\tau_1$ and $\tau_2$ such that $(X, \tau)$ is a Lusin space.*

(2.2) *There are strict subdivisions $\{(\mathscr{C}_n, p_n, \varphi_n)\}$ and $\{\mathscr{D}_n, q_n, \psi_n)\}$ respectively for the spaces $(X, \tau_1)$ and $(X, \tau_2)$ satisfying: For every pair of points $x$, $y$ in $X$, $x \neq y$, there are disjoint sets $\varphi_m(c_m)$ and $\psi_n(d_n)$ such that $x$ is in $\varphi_m(c_m)$ and $y$ is in $\psi_n(d_n)$.*

(2.3) *There is a common strict subdivision for the two spaces $(X, \tau_1)$ and $(X, \tau_2)$.*

PROOF. Clearly (2.3)$\Rightarrow$(2.2). Now, suppose there are subdivisions satisfying the condition (2.2). Let $\tau_1'$ and $\tau_2'$ be the topologies associated with these subdivisions. Clearly $(X \times X, \tau_1' \times \tau_2')$ is a Lusin space and for every $(x, y)$, $x \neq y$, there is a $\tau_1' \times \tau_2'$ open set of the form $\varphi_m(c_m) \times \psi_n(d_n)$ which contains $(x, y)$ and does not intersect the diagonal $\Delta$. It follows that $\Delta$ is $\tau_1' \times \tau_2'$ closed and is hence a Lusin space with the induced topology.

However, $\Delta$ with this topology is homeomorphic to $X$ with the $\sup(\tau_1, \tau_2)$-topology. This shows that (2.2)$\Rightarrow$(2.1). Finally, if (2.1) is satisfied then any strict subdivision of a topology finer than both $\tau_1$ and $\tau_2$ is also a common strict subdivision for both $\tau_1$ and $\tau_2$. The proof is complete.

We have the following result for Suslin spaces with analogous conditions.

THEOREM 3. *Let $\tau_1$ and $\tau_2$ be two topologies on a set $X$ such that $(X, \tau_1)$ and $(X, \tau_2)$ are Suslin spaces. Let there be subdivisions $\{(\mathscr{C}_n, p_n, \varphi_n)\}$ for $(X, \tau_1)$ and $\{(\mathscr{D}_m, q_m, \psi_m)\}$ for $(X, \tau_2)$ such that given $x$, $y$ in $X$, $x \neq y$, there are disjoint sets $\varphi_m(c_m)$ containing $x$ and $\psi_n(d_n)$ containing $y$. Then $X$ with the $\sup(\tau_1, \tau_2)$-topology is a Radon space; that is, every finite Borel measure on this space is Radon.*

PROOF. Let $\tau_1'$ and $\tau_2'$ be the topologies associated with the subdivisions. We observe that $\tau_1'$ (resp. $\tau_2'$) is finer than $\tau_1$ (resp. $\tau_2$) and both the topologies $\tau_1'$ and $\tau_2'$ have countable bases for open sets. Also, as in the proof of the earlier theorem, the given condition on the subdivision implies that the diagonal $\Delta$ is $\tau_1' \times \tau_2'$ closed in $X \times X$. Let $\mathscr{B}(Y, \tau)$ denote the $\sigma$-algebra of Borel subsets of any topological space $(Y, \tau)$ and $\mathscr{M}(Y, \tau)$ the $\sigma$-algebra of all subsets of $Y$ that are measurable for every finite measure on $\mathscr{B}(Y, \tau)$. Now, every Radon space is universally measurable as a subspace of any topological space and this is in particular true for Suslin spaces. We conclude that

$$\mathscr{B}(X, \tau_1) \subset \mathscr{B}(X, \tau_1') \subset \mathscr{M}(X, \tau_1),$$

and

$$\mathscr{B}(X, \tau_2) \subset \mathscr{B}(X, \tau_2') \subset \mathscr{M}(X, \tau_2).$$

Let $A_k$ be in $\mathscr{M}(X, \tau_k)$ for $k=1, 2$. Then $A_k$ with the induced $\tau_k$-topology is Radon and further every $\tau_k$-compact subset $E_k$ of $A_k$ is metrizable since $E_k$ is a compact subset of the Suslin space $(X, \tau_k)$. This shows that $A_1 \times A_2$ with $\tau_1 \times \tau_2$-topology is Radon [**1**]. We deduce easily that

$$\mathscr{M}(X, \tau_1) \otimes \mathscr{M}(X, \tau_2)$$

is contained in $\mathscr{M}[X \times X, \tau_1 \times \tau_2]$. Further, since $(X, \tau_1')$ and $(X, \tau_2')$ both have countable basis for open sets we conclude that

$$\mathscr{B}(X, \tau_1') \otimes \mathscr{B}(X, \tau_2') = \mathscr{B}(X \times X, \tau_1' \times \tau_2').$$

It follows that

$$\Delta \in \mathscr{B}(X \times X, \tau_1' \times \tau_2') \subset \mathscr{M}(X \times X, \tau_1 \times \tau_2).$$

Hence, $\Delta$ is measurable for every finite Borel measure on $(X \times X, \tau_1 \times \tau_2)$. However, this latter space is Radon and it follows that $\Delta$ with the induced

$\tau_1 \times \tau_2$-topology is a Radon space [**1**]. Since $X$ with $\sup(\tau_1, \tau_2)$-topology is homeomorphic to $\Delta$ with the induced $\tau_1 \times \tau_2$-topology, we conclude that $X$ with the $\sup(\tau_1, \tau_2)$-topology is a Radon space, completing the proof.

## Bibliography

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