ON POLYNOMIALS SATISFYING
A TURÁN TYPE INEQUALITY

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Abstract. For Legendre polynomials \( P_n(x) \), P. Turán has established the inequality
\[
\Delta_n(x) = P_n^2(x) - P_{n+1}(x)P_{n-1}(x) \geq 0, \quad -1 \leq x \leq 1, \quad n \geq 1,
\]
with equality only for \( x = \pm 1 \). This inequality has generated considerable interest, and analogous inequalities have been extended to various classes of polynomials: ultraspherical, Laguerre, Hermite, and a class of Jacobi polynomials. Our purpose here is to determine necessary and sufficient conditions for a general class of polynomials to satisfy a Turán type inequality and to characterize the generating functions of such a class.

1. Introduction. In 1948, Szegő [12] called attention to the following remarkable inequality of P. Turán for Legendre polynomials \( P_n(x) \):
\[
\Delta_n(x) = P_n^2(x) - P_{n+1}(x)P_{n-1}(x) = 0, \quad -1 \leq x \leq 1, \quad n \geq 1,
\]
with equality only for \( x = \pm 1 \). This inequality has generated considerable interest (see, e.g., [4] and [10]). Turán’s proof and three additional proofs of (1.1) were given by Szegő [12], who also extended the result to ultraspherical, Laguerre, and Hermite polynomials. More recently, Gasper [2] proved the analogue of (1.1) for a class of Jacobi polynomials. Our purpose here is to determine necessary and sufficient conditions for a general class of polynomials to satisfy a Turán type inequality and to characterize the generating functions of such a class.

Let \( \{a_k\}_{k=0}^{\infty} \) be a sequence of real numbers with \( a_0 = 1 \), let
\[
g_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k,
\]
and let
\[
\Delta_n(x) = g_n^2(x) - g_{n+1}(x)g_{n-1}(x), \quad n \geq 1.
\]
If, for each \( x \), \(-\infty < x < \infty\), either

\[
\Delta_n(x) > 0, \quad n \geq 1, \quad \text{or} \quad \Delta_n(x) = 0, \quad n \geq 1,
\]

then we shall say that the sequence \( \{g_n\}_{n=0}^{\infty} \) satisfies a Turán type inequality.

J. L. Burchnall [1] showed that if \( g_n(x) \) has real, simple zeros for \( n \geq 1 \), then \( \{g_n\} \) satisfies condition (T). In addition, it is easy to see that if

\[
g_n(x) = (1 + ax^n)^n, \quad n \geq 1,
\]

then \( \Delta_n(x) \equiv 0, \quad n \geq 1 \), so that the sequence \( \{g_n\} \) defined by (1.3) trivially satisfies condition (T).

Now it is natural to inquire whether there are other examples of polynomials of the form (1.2) which satisfy a Turán type inequality. By way of an answer to this question, we shall show that provided the coefficients \( \{a_k\} \) satisfy a mild restriction, the two sequences of polynomials mentioned above are the only sequences which satisfy a Turán type inequality, that is, satisfy condition (T). Indeed, if \( \{g_n\} \) is a sequence of polynomials defined by (1.2), then we have

**Theorem 1.** If \( \{g_n\} \) satisfies condition (T) and if \( \Delta_n(0) = 0 \) for some \( \xi \neq 0, \quad n \geq 1 \), then

\[
g_n(x) = (1 + ax^n)^n, \quad n \geq 1.
\]

**Theorem 2.** If \( \Delta_n(x) > 0 \) for all \( x \neq 0, \quad n \geq 1 \), and if the sequence of coefficients \( \{a_k\} \) satisfies the condition

\[
a_{k-1}a_{k+1} < 0 \quad \text{whenever} \quad a_k = 0,
\]

then \( g_n(x) \) has real, simple zeros for \( n \geq 1 \).

We remark that it is not difficult to construct examples which show that Theorem 2 is false if condition (1.4) is omitted.

2. Proof of Theorem 1. If \( \Delta_n(0) = 0 \) for some \( \xi \neq 0, \quad n \geq 1 \), then, in particular, \( \Delta_1(0) = (a_1^2 - a_0a_2)^{\frac{\xi^2}{2}} = 0 \). Hence, \( a_1^2 - a_0a_2 = 0 \) and a fortiori \( \Delta_1(x) = (a_1^2 - a_0a_2)x^2 \equiv 0 \). But then, in view of condition (T),

\[
\Delta_n(x) \equiv 0, \quad n = 1, 2, \ldots.
\]

Now \( \Delta_n(x) \) is a polynomial of degree \( 2n \) with leading coefficient \( a_n^2 - a_{n-1}a_{n+1} \), so (2.1) implies that

\[
a_n^2 - a_{n-1}a_{n+1} = 0, \quad n = 1, 2, \ldots.
\]

Thus, it follows from (2.2) and an easy induction argument that \( a_n = a_1^n, \quad n = 1, 2, \ldots \). Hence,

\[
g_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k = \sum_{k=0}^{n} \binom{n}{k} a_1^k x^k = (1 + a_1 x)^n.
\]
3. **Proof of Theorem 2.** The proof of Theorem 2 depends upon an algebraic rule, which Pólya [6, p. 21] credits to de Gua, and a lemma.

**de Gua’s rule.** A polynomial \( f(x) \) with real coefficients has real, simple zeros only, if its derivatives \( f'(x), f''(x), \cdots, f^{(n)}(x), \cdots \) have the property:

If \( \xi \) is real and \( f^{(n)}(\xi) = 0 \), then \( f^{(n-1)}(\xi)f^{(n+1)}(\xi) < 0 \).

**Lemma.** Under the hypothesis of Theorem 2,

\[
a^2_n - a_{n-1}a_{n+1} > 0, \quad n = 1, 2, \cdots.
\]

**Proof.** The proof will be by induction. First observe that, by hypothesis, \( \Delta_1(x) = (a_1^2 - a_0a_2)x^2 > 0, \, x \neq 0 \), and hence,

\[
(3.1) \quad a_1^2 - a_0a_2 > 0.
\]

Now suppose

\[
(3.2) \quad a_1^2 - a_0a_2 > 0, \quad a_2^2 - a_1a_3 > 0, \quad \cdots, \quad a_{n-1}^2 - a_{n-2}a_n > 0.
\]

To show that \( a_n^2 - a_{n-1}a_{n+1} > 0 \), note that

\[
\Delta_n(x) = \sum_{k=2}^{2n} c_k x^k,
\]

where

\[
c_{2n} = a_n^2 - a_{n-1}a_{n+1} \quad \text{and} \quad c_{2n-1} = (n - 1)(a_n a_{n-1} - a_{n-2}a_{n+1}).
\]

Thus, the hypothesis \( \Delta_n(x) > 0, \, x \neq 0 \), implies \( a_n^2 - a_{n-1}a_{n+1} \geq 0 \) and

\[
(3.3) \quad a_na_{n-1} = a_{n-2}a_{n+1} \quad \text{whenever} \quad a_n^2 - a_{n-1}a_{n+1} = 0.
\]

Now if \( a_n = 0 \), then it follows from (1.4) that \( a_n^2 - a_{n-1}a_{n+1} > 0 \). If, on the other hand, \( a_n \neq 0 \) and \( a_n^2 - a_{n-1}a_{n+1} = 0 \), then (3.3) implies \( a_na_{n-1} = a_{n-2}a_{n+1} \). Consequently, it follows that \( a_n^2 - a_{n-2}a_n = 0 \). This contradicts (3.2) and thus, the induction is complete.

We now proceed with the proof of Theorem 2. First, set

\[
(3.4) \quad P_n(x) = (1/n!)x^ng_n(x^{-1})
\]

and observe that

\[
(3.5) \quad P_n'(x) = P_{n-1}(x).
\]

Next, express \( x^{2n}\Delta_n(x^{-1}) \) in terms of the polynomials defined by (3.4) to obtain

\[
(3.6) \quad x^{2n}\Delta_n(x^{-1}) = (n + 1)!(n - 1)! \left[ \frac{n}{n + 1} P_n^2(x) - P_{n-1}(x)P_{n+1}(x) \right].
\]
Since by hypothesis $\Delta_n(x) > 0$ for $x \neq 0$, $n \geq 1$, (3.6) implies

$$\sigma_n(x) = \frac{n}{n+1} P_n(x) - P_{n-1}(x)P_{n+1}(x) > 0, \quad x \neq 0, \quad n \geq 1. \quad (3.7)$$

Moreover, the preceding lemma implies

$$\alpha_n = -n(n+1) & n=1, \quad (3.8)$$

Thus, by (3.7) and (3.8), we have

$$\sigma_n(x) > 0, \quad -\infty < x < \infty, \quad n \geq 1. \quad (3.9)$$

Now suppose that $P_n^{(k)}(\xi) = 0$. Then by (3.5) and (3.9), we have

$$0 < \sigma_{n-k}(\xi) = \frac{n-k}{n-k+1} P_n^{(k)}(\xi) - P_{n-k+1}(\xi)P_{n-k-1}(\xi)$$

$$= \frac{n-k}{n-k+1} [P_n^{(k)}(\xi)]^2 - P_n^{(k-1)}(\xi)P_n^{(k+1)}(\xi)$$

$$= -P_n^{(k-1)}(\xi)P_n^{(k+1)}(\xi).$$

Thus,

$$P_n^{(k-1)}(\xi)P_n^{(k+1)}(\xi) < 0, \quad k = 1, \ldots, n - 1,$$

and de Gua's rule implies $P_n(x)$ has real, simple zeros for $n \geq 1$. Since $g_n(x) = n! x^n P_n(x^{-1})$, it follows that $g_n(x)$ has real, simple zeros for $n \geq 1$.

Theorem 2 has the following immediate but interesting

COROLLARY. Let $\Delta_n(x)$ and $\{a_n\}$ satisfy the hypothesis of Theorem 2 and set

$$g_{n,p}(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k, \quad n \geq 1, \quad p \geq 1.$$  

Then $g_{n,p}(x)$ has real, simple zeros and, for every $p \geq 1$, the sequence $\{g_{n,p}\}$ satisfies condition (T).

PROOF. Since

$$g_{n,p}(x) = \frac{p!}{(n+p)!} g_{n+p}^{(p)}(x)$$

and $g_{n+p}(x)$ has real, simple zeros, Rolle's theorem implies that $g_{n,p}(x)$ has real, simple zeros. The assertion that $\{g_{n,p}\}$ satisfies condition (T), for $p \geq 1$, then follows from Burchnall's result mentioned in the Introduction.
4. The generating functions of polynomials satisfying condition (T).

Suppose

\[ f(z) = \sum_{k=0}^{\infty} \frac{a_k z^k}{k!} \quad (a_0 = 1) \]

is holomorphic in a neighborhood of the origin. It is well known (see, e.g., [8]) that the sequence of polynomials \( \{g_n\} \), defined by (1.2), is generated by \( e^{zf}(xz) \), that is, \( e^{zf}(xz) = \sum_{n=0}^{\infty} g_n(x) z^n/n! \), while the sequence of polynomials \( \{P_n\} \), defined by (3.4), is generated by \( e^{z^2 f}(z) \), that is, \( e^{z^2 f}(z) = \sum_{n=0}^{\infty} P_n(x) z^n \). (The polynomials \( P_n(x) \) are called Appell polynomials.)

Of special interest is the case when \( f(z) \) is of the form

\[ f(z) = e^{-z^{2}+\beta z} \prod_{n=1}^{\infty} (1 - z/n) e^{z/z_n} \]

where \( \gamma \geq 0, \beta, z_n \) are real and \( \sum_{n=1}^{\infty} z_n^{-2} < \infty \).

We shall say that an entire function \( f(z) \) of the form (4.2) belongs to the class \( \mathcal{L} - \mathcal{P} \) (Laguerre-Pólya) and we shall write \( f(z) \in \mathcal{L} - \mathcal{P} \).

If \( f(z) \in \mathcal{L} - \mathcal{P} \) is given by (4.1), then it is well known [7, p. 110] that, for \( n \geq 1 \), \( g_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k \) has only real zeros. Consequently, it follows ([3, §4.3] or [9, p. 76]) that

\[ a_k^2 - a_{k-1} a_{k+1} > 0, \quad k \geq 1, \quad \text{or} \quad a_k^2 - a_{k-1} a_{k+1} = 0, \quad k \geq 1. \]

(Note that the second condition in (4.3) implies that \( f(z) = e^{a_1 z} \).) Since \( f(z) \in \mathcal{L} - \mathcal{P} \) clearly implies \( e^{zf}(xz) \in \mathcal{L} - \mathcal{P} \) and \( e^{z^2 f}(z) \in \mathcal{L} - \mathcal{P} \) for every \( x, -\infty < x < \infty \), the following proposition is a consequence of (4.3).

**Proposition 1.** Let \( f(z) \) be given by (4.1). If \( f(z) \in \mathcal{L} - \mathcal{P} \), then the polynomial sequences \( \{g_n\} \) and \( \{n! P_n\} \) generated by \( e^{zf}(xz) \) and \( e^{z^2 f}(z) \) respectively, satisfy condition (T).

Conversely, as a consequence of Theorem 2, we have

**Proposition 2.** If \( \{a_k\}_{k=0}^{\infty}, a_0 = 1 \), is a sequence of real numbers which satisfies (1.4) and if the sequence \( \{g_n\} \) defined by (1.2) satisfies condition (T) then the function

\[ f(z) = \sum_{k=0}^{\infty} \frac{a_k z^k}{k!} \]

belongs to the class \( \mathcal{L} - \mathcal{P} \).

\footnote{Szegő [12] used this condition to show that many of the classical polynomials satisfy a Turán type inequality.}
Proof. By Theorems 1 and 2, \( g_n(x) \), \( n \geq 1 \), has only real zeros, and hence, the polynomial \( G_n(z) \), \( n \geq 1 \), where

\[
G_n(z) = g_n\left(\frac{z}{n}\right) = \sum_{k=0}^{n} a_k \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) z^k
\]

has only real zeros. Furthermore, since

\[
G_n(0) = 1, \quad |G_n'(0)| = |a_1| \quad \text{and} \quad |G_n''(0)| \leq \frac{1}{2} |a_2|,
\]

it follows (see, e.g., Szász [11]) that \( \{G_n(z)\} \) is a normal family. Now \( f(z) \) is clearly the unique limit function of the sequence \( \{G_n(z)\} \); thus, \( \{G_n(z)\} \) converges uniformly to \( f(z) \) on every compact subset of the plane. Since \( G_n(z) \), \( n \geq 1 \), has only real zeros, a classical result of Pólya [5] implies that \( f(z) \in \mathcal{L}_-\mathcal{P} \).

References


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