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A NOTE ON SEMITOPOLOGICAL CLASSES

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ABSTRACT. This paper shows that semitopological classes are subsemilattices of the lattice of topologies, and gives a new characterization for the finest topology in the semitopological class.

Introduction. In [**4**] Levine defined a set, A, to be *semiopen* if there is some open set U so that $U \subset A \subset c(U)$, where $c(\ )$ denotes closure in the topological space. In [**1**] it was shown that if (X, τ) is a topological space, there is a finest topology [we shall call it $F(\tau)$] so that the semiopen sets are the same as for τ. If X is a set of points, let $T(X)$ be the lattice of topologies on X. If $\tau \in T(X)$, let $[\tau]$ denote the equivalence class of all topologies which have the same semiopen sets as τ. $[\tau]$ is called a *semitopological class* of topologies on X. The object of this note is to show that if $\tau \in T(X)$, $[\tau]$ is a subsemilattice of $T(X)$ with respect to the usual join operation on topologies, and to give a new characterization for $F(\tau)$.

1. **Semitopological classes are subsemilattices of the lattice of topologies.** In [**1**] a set was defined to be *semiclosed* if its complement is semiopen, and *semiclosure* and *semi-interior* were defined in a manner analogous to the definitions of closure and interior.

LEMMA 1.1. *If (X, τ) is a topological space, and if $c(\ )$ and $i(\ )$ denote the closure and interior, respectively, in (X, τ) while $c^*(\ )$ and $i^*(\ )$ denote the closure and interior in $(X, F(\tau))$, and $sc(\ )$ and $si(\ )$ denote the semiclosure and semi-interior in both, then if $A \subset X$, $i^*(c(A)) \subset sc(A)$.*

PROOF. If $O \in F(\tau)$ so that $O \subset c(A)$, then consider $O \cap (X - sc(A))$. By Theorem 1.9 of [**1**], the intersection of an open set and a semiopen set is semiopen. Since $sc(A)$ is semiclosed, $(X - sc(A))$ is semiopen; therefore $O \cap (X - sc(A))$ is semiopen in $(X, F(\tau))$. Consequently, $O \cap (X - sc(A)) = O - sc(A)$ is semiopen in (X, τ). Now since $O \subset c(A)$, we have

$$O - sc(A) \subset c(A) - sc(A) \subset c(A) - A.$$

By Theorem 1.14 of [**1**], $si(c(A) - A) = \varnothing$. Therefore, since $O - sc(A)$ is semiopen, $O - sc(A) = \varnothing$, so that $O \subset sc(A)$.

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Consequently, since any element of $F(\tau)$ which is a subset of $c(A)$ must also be a subset of $sc(A)$, $i^*(c(A)) \subset sc(A)$.

LEMMA 1.2. *If (X, τ) is a topological space, and if $c(\ )$, $i(\ )$, $c^*(\ )$, and $i^*(\ )$ are as in Lemma* 1.1, *then if $A \subset X$,*

$$i(c^*(A)) = i(c(A)).$$

PROOF. Since $F(\tau)$ is finer than τ, it is clear that $c^*(A) \subset c(A)$ so that $i(c^*(A)) \subset i(c(A))$. Thus it only remains to show that $i(c(A)) \subset i(c^*(A))$. If $O \subset c(A)$ and $O \in \tau$, then $O \in F(\tau)$ and $O \cap (X - c^*(A)) = O - c^*(A)$ is in $F(\tau)$. Since $O - c^*(A)$ is in $F(\tau)$, $O - c^*(A)$ is semiopen with respect to each of τ and $F(\tau)$. Since $O \subset c(A)$, we have

$$O - c^*(A) \subset c(A) - c^*(A) \subset c(A) - A;$$

and, as in Lemma 1.1, $si(c(A) - A) = \varnothing$, so that $O - c^*(A) = \varnothing$. Thus $O \subset c^*(A)$. Consequently, since each element of τ which is a subset of $c(A)$ is also a subset of $c^*(A)$, $i(c(A)) \subset i(c^*(A))$. Hence we have $i(c^*(A)) = i(c(A))$.

In [1] an example is given of two topologies τ and σ on a set X such that σ is a proper subset of τ, while $SO(X, \tau)$ [the collection of all semiopen subsets of X with respect to τ] is a proper subset of $SO(X, \sigma)$. However, based on the last two lemmas, we do have the following theorem.

THEOREM 1. *If (X, τ) is a topological space, and if $\tau \subset \sigma \subset F(\tau)$, then $\sigma \in [\tau]$.*

PROOF. Let $c(\ )$ and $i(\ )$ denote the closure and interior, respectively, in τ. Let $c^+(\ )$ and $i^+(\ )$ denote the closure and interior in σ, and let $c^*(\ )$ and $i^*(\ )$ denote the closure and interior in $F(\tau)$. In [1] it was shown that if a presemiclosure $(\ )_c$ is *consistent* with a closure operator $(\ )^c$, that is;

(1) if $A \subset X$, $(A^c)^i \subset A_c$ where $(\ )^i$ is the interior induced by $(\ )^c$, and

(2) if $(A^c)^i \subset A$, then $A_c = A$,

then $(\ )_c$ is the semiclosure in the topology generated by $(\ )^c$. Consequently, we need only show that the semiclosure in (X, τ), denoted by $sc(\ )$, is consistent with the closure in (X, σ).

First we want to show $i^+(c^+(A)) \subset sc(A)$. Since σ is finer than τ, $c^+(A) \subset c(A)$, so that $i^+(c^+(A)) \subset i^+(c(A))$. Furthermore, since $F(\tau)$ is finer than σ, $i^+(c(A)) \subset i^*(c(A))$. Thus $i^+(c^+(A)) \subset i^*(c(A))$. But by Lemma 1.1, $i^*(c(A)) \subset sc(A)$, so that $i^+(c^+(A)) \subset sc(A)$. Second, we want to show that if $i^+(c^+(A)) \subset A$, then $A = sc(A)$. Since σ is finer than τ, $i(c^+(A)) \subset i^+(c^+(A))$, and since σ is coarser than $F(\tau)$, $c^*(A) \subset c^+(A)$ so that $i(c^*(A)) \subset i(c^+(A))$. Therefore we have $i(c^*(A)) \subset i^+(c^+(A))$. By Lemma 1.2, $i(c^*(A)) = i(c(A))$.

Consequently, $i(c(A)) \subset i^+(c^+(A))$, so that if $i^+(c^+(A)) \subset A$, then $i(c(A)) \subset A$, and since $sc(\ )$ is the semiclosure with respect to τ, $sc(A)=A$ by Theorem 1.12 of [1]. Thus by Theorem 2.5 of [1], the semiclosed (and consequently the semiopen sets) in (X, σ) are precisely those in (X, τ). Consequently $\sigma \in [\tau]$.

COROLLARY 1.1. *If (X, τ) and (X, σ) are topological spaces with the same semiopen sets, then if $\tau \vee \sigma$ is the usual join in the lattice of topologies $T(X)$, then $\tau \vee \sigma \in [\tau] = [\sigma]$.*

PROOF. Since $F(\tau) \supset \tau$ and $F(\tau) \supset \sigma$ then $F(\tau) \supset \tau \vee \sigma \supset \tau$, and by Theorem 1.3, $\tau \vee \sigma \in [\tau]$.

Example 2.1 of [3] shows that if (X, τ) and (X, σ) are topological spaces such that $SO(X, \tau) = SO(X, \sigma)$, it is not necessarily the case that $\tau \cap \sigma \in [\tau]$.

In this section, we have seen that with the usual join operation for topologies, semitopological classes are subsemilattices of the lattice of all topologies on X. Furthermore, these semilattices all have maximal elements.

2. **A new characterization of $F(\tau)$.** It was shown in [3] that the finest topology on the set of real numbers for which the semiopen sets are those of the usual topology is the collection $\{O-N \mid O$ is open in the usual topology and N is nowhere dense in the usual topology$\}$.

THEOREM 2. *If (X, τ) is a topological space and if ν is the collection of all sets which are nowhere dense in (X, τ), then $F(\tau) = \{U-N \mid U \in \tau$ and $N \in \nu\}$.*

PROOF. The only way to find $F(\tau)$ has been to go through the construction process outlined in [1]. Given a semiclosure operator $(\ )_c$, we can construct the closure operator for $F(\tau)$ in the following way. For every subset A there exists a minimal set D_A such that $(A \cup D_A \cup B)_c = (A \cup D_A \cup B_c)$ for all $B \subset X$. [D_A is minimal in the sense that it is a subset of any set satisfying this same condition.] Then defining the closure of A by $A \cup D_A$, we get the closure in $F(\tau)$ [Theorems 2.11 and 2.19 of [1]].

We want to show that $F(\tau) = \{U-N \mid U \in \tau, N \in \nu\}$. That is, we want to show that the sets closed in $F(\tau)$ are $\{K \cup N \mid (X-K) \in \tau, N \in \nu\}$. Consequently the theorem will be proven if we can show that for $A \subset X$, $D_A = \varnothing$ if and only if there exist K, closed in (X, τ), and $N \in \nu$ so that $A = K \cup N$.

Now, first consider any set of the form $K \cup N$ where $(X-K)$ is in τ and $N \in \nu$. In order to show that $D_{K \cup N} = \varnothing$, it is only necessary to show that for any $B \subset X$, $sc(K \cup N \cup B) = [K \cup N \cup sc(B)]$.

Now $sc(K\cup N\cup B)\supset[sc(K)\cup sc(N)\cup sc(B)]$. Furthermore, since K is closed in (X, τ), K is semiclosed so that $sc(K)=K$, and since all nowhere dense sets are semiclosed, $sc(N)=N$. Thus $sc(K\cup N\cup B)\supset[K\cup N\cup sc(B)]$.

Now if it can be demonstrated that $K\cup N\cup sc(B)$ is semiclosed, it will follow that since $[K\cup N\cup B]\subset[K\cup N\cup sc(B)]$, $sc(K\cup N\cup B)\subset[K\cup N\cup sc(B)]$ and we will have $sc(K\cup N\cup B)=[K\cup N\cup sc(B)]$. $K\cup N\cup sc(B)$ is semiclosed if and only if $i(c(K\cup N\cup sc(B)))\subset[K\cup N\cup sc(B)]$. Now

$$\begin{aligned} c(K\cup N\cup sc(B)) &= [c(K)\cup c(N)\cup c(sc(B))] \\ &= [K\cup c(N)\cup c(B)]. \end{aligned}$$

If $W\in\tau$ so that $W\subset[K\cup c(N)\cup c(B)]$, then $W\subset[K\cup c(B)]$, for if not $W\cap(X-(K\cup c(B)))$ is open and nonvoid and a subset of $c(N)$ which contradicts the fact that N is nowhere dense. Thus, $W\subset[K\cup c(B)]$. Furthermore, $W\subset[K\cup sc(B)]$, for otherwise, since $K\cup sc(B)$ is semiclosed, $W\cap(X-(K\cup sc(B)))$ would be semiopen and nonvoid and a subset of $c(B)-sc(B)$. But $c(B)-sc(B)$ is a subset of $c(B)-B$, and $i(c(B)-B)=\varnothing$. Therefore there can be no nonvoid, semiopen subset of $c(B)-sc(B)$. Thus $W\subset[K\cup sc(B)]$. Therefore, since $W\subset[K\cup sc(B)]\subset[K\cup N\cup sc(B)]$, it follows that $i(c(K\cup N\cup sc(B)))\subset[K\cup N\cup sc(B)]$ so that $[K\cup N\cup sc(B)]$ is semiclosed. Thus for any $B\subset X$,

$$[K\cup N\cup sc(B)] = sc(K\cup N\cup B), \quad \text{and } D_{K\cup N}=\varnothing.$$

Now, if $D_G=\varnothing$, G is closed in $F(\tau)$ so that G is semiclosed in both $F(\tau)$ and τ. Since $i(G)$ is open in τ, it is semiopen in τ. Thus by Theorem 1.7 of [2], $c(i(G))$ is semiopen in τ and thus, also in $F(\tau)$. $X-G$ is open in $F(\tau)$ so that $c(i(G))\cap(X-G)=(c(i(G))-G)$ is semiopen in $F(\tau)$. Now since $(c(i(G))-G)$ is semiopen, it must be empty, for otherwise, there would be a nonvoid, open subset of $(c(i(G))-G)$, which is not possible. Thus $c(i(G))\subset G$, and note that $c(i(G))$ is closed in τ. Since G is semiclosed in τ, there is a set H, closed in τ, so that $i(H)\subset G\subset H$. Clearly $i(H)=i(G)$. $H-i(H)$ is nowhere dense in τ and therefore, since

$$(G-c(i(G)))\subset(G-i(G))=(G-i(H))\subset(H-i(H)),$$

$G-c(i(G))$ is nowhere dense in τ. Thus

$$G=c(i(G))\cup(G-c(i(G))),$$

where $c(i(G))$ is closed in (X, τ), and $G-c(i(G))$ is nowhere dense in (X, τ).

REFERENCES

1. S. Gene Crossley and S. K. Hildebrand, *Semi-closure*, Texas J. Sci. **22** (2 & 3) (1971), 99–112.

2. ———, *Semi-closed sets and semi-continuity in topological spaces*, Texas J. Sci. **22** (2 & 3) (1971), 123–126.

3. ———, *Semi-topological properties*, Fund. Math. **74** (1972), 233–254.

4. Norman L. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41. MR **29** #4025.

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