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ON CARMICHAEL'S CONJECTURE

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ABSTRACT. A sufficient condition is given for a natural number x in order that the equation $\varphi(x)=\varphi(y)$ has only the solution $y=x$. It is conjectured that no natural numbers satisfy this sufficient condition.

Denote by $N(m)$ the number of solutions x to the equation $\varphi(x)=m$, where φ is Euler's totient function. R. D. Carmichael [1] conjectured that for every m, $N(m)\neq 1$. V. L. Klee, Jr. [2] proved that if $N(\varphi(x))=1$, then x must necessarily satisfy a stringent set of conditions. In particular, these conditions led Klee to conclude that if $N(\varphi(x))=1$, then both x and $\varphi(x)$ are $>10^{400}$. It is an immediate consequence of Klee's work that if $N(m)=1$, then $m\equiv 0 \pmod{2^{42}}$ and $m\equiv 0 \pmod{3^{47}}$.

It is the purpose of this note to give a sufficient condition on x for $N(\varphi(x))=1$.

THEOREM. *Suppose x is a natural number such that for every prime p, $(p-1)|\varphi(x)$ implies $p^2|x$. Then $N(\varphi(x))=1$.*

If n is a natural number, denote by $S(n)$ the set of primes dividing n. If p is a prime, denote by $v_p(n)$ the exponent (possibly zero) on p in the prime factorization of n. Hence

$$\begin{aligned} v_p(\varphi(n)) &= \sum_{q\in S(n)} v_p(q-1), && \text{if } p\nmid n,\\ &= v_p(n)-1+\sum_{q\in S(n)} v_p(q-1), && \text{if } p\mid n. \end{aligned}$$

Now suppose x satisfies the condition in the theorem, and let y be such that $\varphi(y)=\varphi(x)$. To prove the theorem it will be sufficient to show $y=x$. We first note that if $p\in S(y)$, then $(p-1)|\varphi(y)=\varphi(x)$, so by assumption $p^2|x$. That is, $S(y)\subset S(x)$. Now suppose $p\in S(x)$. Then $(p-1)|\varphi(x)$, so $p^2|x$. If $p\notin S(y)$, then

$$\begin{aligned} v_p(x)-1+\sum_{q\in S(x)} v_p(q-1)\\ = v_p(\varphi(x)) = v_p(\varphi(y)) = \sum_{q\in S(y)} v_p(q-1) \leqq \sum_{q\in S(x)} v_p(q-1) \end{aligned}$$

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(since $S(y) \subset S(x)$), contradicting $v_p(x)-1 \geqq 1$. Hence $p \in S(y)$ and in fact, $S(x)=S(y)$. Now if $p \in S(x)=S(y)$, we have

$$\begin{aligned} v_p(x) &= v_p(\varphi(x)) + 1 - \sum_{q \in S(x)} v_p(q-1) \\ &= v_p(\varphi(y)) + 1 - \sum_{q \in S(y)} v_p(q-1) = v_p(y). \end{aligned}$$

This proves that $x=y$, and hence establishes the theorem.

However, it is likely that no number x exists having the property described in the theorem. Indeed if the following conjecture is true, no such number exists:

CONJECTURE. If $k \geqq 2$, then $(p_k-1) \mid \prod_{i=1}^{k-1} p_i(p_i-1)$, where p_i denotes the ith prime.

If x has the property described in the theorem, then $2^2 \mid x$. Hence if the conjecture is true, then $p_k^2 \mid x$ whenever $p_1^2, p_2^2, \cdots, p_{k-1}^2$ all divide x, and hence x is divisible by every prime.

Suppose there is a prime q such that the smallest prime $p \equiv 1 \pmod{q}$ is also $\equiv 1 \pmod{q^2}$. Then the conjecture fails for $p_k=p$. However we note that conjecture H_2 of Schinzel [3] would deny the existence of such a prime q.

REFERENCES

1. R. D. Carmichael, *Note on Euler's φ-function*, Bull. Amer. Math. Soc. **28** (1922), 109–110.

2. V. L. Klee, Jr., *On a conjecture of Carmichael*, Bull. Amer. Math. Soc. **53** (1947), 1183–1186. MR **9**, 269.

3. A. Schinzel and W. Sierpiński, *Sur certaines hypothèses concernant les nombres premiers*, Acta Arith. **4** (1958), 185–208. MR **21** #4936.

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