SOME PHENOMENA IN HOMOTOPICAL ALGEBRA

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Abstract. In [6] D. G. Quillen developed homotopy theory in categories satisfying certain axioms. He showed that many results in classical homotopy theory (of topological spaces) go through in his axiomatic set-up. The duality observed by Eckmann-Hilton in classical homotopy theory is reflected in the axioms of a model category. In [7] we developed the theory of numerical invariants like the Lusternik-Schnirelmann category and cocategory etc. for such model categories and in [8] we dealt with applications of this theory to injective and projective homotopy theory of modules as developed by Hilton [2], [3, Chapter 13].

Contrary to the general expectations there are many aspects of classical homotopy theory which cannot be carried over to Quillen's axiomatic set-up. This paper deals with some of these phenomena.

Introduction. For any topological group \( G \) it is well known [1], [5], that there exists a principal fibre space \( E_G \to \pi_0 B_G \) with group \( G \) and total space \( E_G \) contractible. This suggests the following question. Suppose \( M \) is a group object in a model category \( \mathcal{C} \) in the sense of Quillen [6]. Does there exist a fibration \( E \to \pi B \) in \( \mathcal{C} \) with the property that \( E \) is contractible (i.e. to say \( \pi(Q(E), Q(E)) = 0 \) following the notation of Quillen [6]) with fibre of \( p \) isomorphic to \( M \)? We will give examples to show that, in general, this is false. Also we will illustrate that, given a cogroup object \( H \) in a model category \( \mathcal{C} \), there need not exist a cofibration \( A \to q E \) in \( \mathcal{C} \) with \( E \) contractible and cofibre of \( q \) isomorphic to \( H \).

Actually it will turn out that the two model categories \( \mathcal{C} \) and \( \mathcal{T} \) that we mention in this connection (§1) will have the following additional properties.

(i) All the objects are simultaneously group objects and cogroup objects.

(ii) For every object \( A \) both \( \Sigma A \) and \( \Omega A \) are contractible.

It can easily be shown that in the category \( \mathcal{T} \) of topological spaces if \( G \) is a group object with \( \Sigma G \) contractible then \( G \) itself is contractible.

In §2 we characterise all CW-complexes \( X \) with the property that \( \Sigma X \) is contractible. They turn out to be "Moore CW-complexes" \( M(\pi, 1) \)

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for groups \( \pi \) satisfying \( H_1(\pi) = 0 = H_2(\pi) \). On the other hand, if \( X \) is a 0-connected CW-complex with \( \Omega X \) contractible, then \( X \) itself is contractible.

1. The model categories \( \mathcal{C} \) and \( \mathcal{F} \). Let \( \mathcal{C} \) denote the category of all modules over a Dedekind domain \( A \). Defining cofibrations, weak equivalences and fibrations to be respectively monomorphisms, \( i \)-homotopy equivalences in the sense of Hilton [2] and maps satisfying the lifting property (L.P.) below, the author showed in [8] that \( \mathcal{C} \) is a model category satisfying the axioms \( M_0 \) to \( M_5 \) of Quillen [6].

(L.P.) A map \( p: E \rightarrow B \) in \( \mathcal{C} \) satisfies (L.P.) if given any \( f: J \rightarrow B \) with \( J \) injective there exists a lift \( g: J \rightarrow E \) of \( f \) (i.e. \( pg = f \)).

Let \( \mathcal{F} \) be the category of finitely generated modules over a Principal Ideal Domain (PID). Defining fibrations, weak equivalences and cofibrations to be respectively epimorphisms, \( p \)-homotopy equivalences in the sense of Hilton [2] and maps satisfying the extension property (E.P.) mentioned below it was shown in [8] that \( \mathcal{F} \) is a model category in the sense of Quillen.

(E.P.) A map \( q: A \rightarrow E \) is said to have the (E.P.) if given any finitely generated free \( A \)-module \( F \) and any map \( \alpha: A \rightarrow F \) there exists a map \( \beta: E \rightarrow F \) satisfying \( \beta q = \alpha \).

It is clear that for any \( M \) in \( \mathcal{C} \) (resp. \( \mathcal{F} \)) \( M \times M \rightarrow M, M \rightarrow M \) defined by \( \mu(x, y) = x + y, \sigma(x) = -x \) make \( M \) into a group object in \( \mathcal{C} \) (resp. \( \mathcal{F} \)) with \( \mu \) as the multiplication, \( \sigma \) as the inversion and \( 0: M \rightarrow M \) as the unit. Similarly, \( M \rightarrow M \oplus M \) given by \( \nu(x) = (x, x) \) makes \( M \) into a cogroup object in \( \mathcal{C} \) (resp. \( \mathcal{F} \)) with \( \sigma \) as the inversion and \( M \rightarrow 0 \) as the co-unit. The following were proved in [8].

(1) In \( \mathcal{C} \) as well as \( \mathcal{F} \) all the objects are fibrant and cofibrant.

(2) An object \( M \) of \( \mathcal{C} \) (resp. \( \mathcal{F} \)) is contractible if and only if \( M \) is injective (respectively free).

(3) For any \( M \) in \( \mathcal{C} \) as well as \( \mathcal{F} \) both \( \Sigma M \) and \( \Omega M \) are contractible.

(4) For \( M \) in \( \mathcal{C} \) or \( \mathcal{F} \)

(a) \( \text{Ind Cat } M = 0 = \text{Cocat } M \) if and only if \( M \) is contractible.

(b) \( \text{Ind Cat } M = \infty = \text{Cocat } M \) whenever \( M \) is not contractible.

**Proposition 1.1.** Let \( M \in \mathcal{C} \) (resp. \( \mathcal{F} \)).

(i) If there exists a fibration \( p: E \rightarrow \mathcal{p}B \) with \( E \) contractible and fibre of \( p \) isomorphic to \( M \), then \( M \) itself is contractible.

(ii) If there exists a cofibration \( q: A \rightarrow E \) with \( E \) contractible and cofibre of \( q \) isomorphic to \( M \), then \( M \) itself is contractible.

**Proof.** If there exists a fibration \( E \rightarrow \mathcal{p}B \) with \( E \) contractible and fibre of \( p \) isomorphic to \( M \) then, from the definition of Cocat \( M \), we see...
that $\text{Cocat } M \leq 1$. Then 4(b) implies $M$ is contractible. This proves (i). The proof of (ii) is exactly dual and hence omitted.

2. Contractibility of $\Sigma X$. We now consider the category $\mathcal{F}_\ast$ of pointed topological spaces. Unless otherwise mentioned the homology groups we consider are the singular homology groups.

**Proposition 2.1.** Let $X$ be a topological space which is of the homotopy type of $\Omega Y$ for some $Y$. Then $\Sigma X$ is contractible if and only if $X$ itself is contractible.

**Proof.** When $X$ is contractible clearly $\Sigma X$ also is. Assume $\Sigma X$ contractible. Let $f : X \to \Omega Y$ be a homotopy equivalence. Then

$$[X, X] \overset{f_*}{\to} [X, \Omega Y] \cong [\Sigma X, Y] = 0$$

since $\Sigma X$ is contractible. Thus $[X, X] = 0$, and $X$ is contractible.

**Corollary 2.2.** Let $G$ be a topological group. Then $SG$ is contractible if and only if $G$ itself is.

**Proof.** It is known that $G$ is of the homotopy type of $\Omega B_G$ where $B_G$ is a classifying space for $G$.

**Remark 2.3.** When $G$ is a group object in $\mathcal{F}_\ast$ the above corollary asserts that $\Sigma G$ is contractible if and only if $G$ itself is. Consider the model categories $\mathcal{C}$ and $\mathcal{F}$ introduced in §1. All the objects in $\mathcal{C}$ (or $\mathcal{F}$) are group objects; for any object $M$ both $\Sigma M$ and $\Omega M$ are contractible. By taking the base ring $\Lambda$ to be the ring $Z$ of integers we see immediately that not all $M$ in $\mathcal{C}$ (resp. $\mathcal{F}$) are contractible.

**Definition 2.4.** Given any group $\pi$ not necessarily abelian we call a space $X$ a "Moore space" of type $(\pi, 1)$; if $X$ is arcwise connected, $\pi_1(X) \cong \pi$ and $H_j(X) = 0$ for $j \geq 2$.

This definition differs from the one given in [9] in only one respect. We allow $\pi$ to be nonabelian. We denote a Moore space of type $(\pi, 1)$ by $M(\pi, 1)$. Let $H_i(\pi)$ denote the $i$th homology group of the group $\pi$ with coefficients in $Z$ (with trivial $\pi$-operators). The following is proved in [9].

**Proposition 2.5.** A Moore space $M(\pi, 1)$ exists if and only if $H_2(\pi) = 0$.

The proof given in [9] is valid even if $\pi$ is not abelian. When $H_2(\pi) = 0$ the construction in [9] actually gives an $M(\pi, 1)$ CW-complex.

**Proposition 2.6.** Let $X$ be a CW-complex. Then $\Sigma X$ is contractible if and only if $X$ is an $M(\pi, 1)$ complex with $H_1(\pi) = 0 = H_2(\pi)$.

**Proof.** Assume $\Sigma X$ contractible. If $\alpha$ is the cardinality of the set of arc components of $X$ then $H_1(\Sigma X)$ is free abelian of rank $\alpha - 1$. Since
$H_1(\Sigma X) = 0$ we see that $\alpha = 1$. Thus $X$ is 0-connected. Let $\pi$ denote $\pi_1(X)$. Then from $0 = H_{j+1}(\Sigma X) \simeq H_j(X)$ for $j \geq 1$ we see that $H_1(X) \simeq \pi/\{1\} \simeq H_1(\pi) = 0$ and $H_j(X) = 0$ for $j \geq 2$. Hence, $X$ is an $M(\pi, 1)$ complex with $H_1(\pi) = 0$. From Proposition 2.5 we get $H_2(\pi) = 0$.

Conversely, assume $X$ is an $M(\pi, 1)$ CW-complex with $H_1(\pi) = 0$. $\Sigma X$ is simply connected (Van Kampen theorem). From $H_{j+1}(\Sigma X) = H_j(X)$ for $j \geq 1$, $H_j(X) = 0$ for $j \geq 2$ and $H_1(X) \simeq \pi/\{1\} \simeq H_1(\pi) = 0$ we immediately see that $H_i(\Sigma X) = 0$ for all $i \geq 1$. By J. H. C. Whitehead $\Sigma X$ is contractible.

**Remark 2.7.** Finitely presentable groups $\pi$ satisfying $H_1(\pi) = 0 = H_2(\pi)$ are known to be the groups which occur as the fundamental groups of smooth homology $n$-spheres ($n \geq 5$) [4]. There are many such nontrivial groups.

Thus there are noncontractible CW-complexes $X$ with $\Sigma X$ contractible.

### 3. Contractibility of $\Omega X$.

**Lemma 3.1.** Suppose $X$ is of the homotopy type of a 0-connected CW-complex. Then $\Omega X$ is contractible if and only if $X$ is.

This is an immediate consequence of the relation $\pi_i(\Omega X) \simeq \pi_{i+1}(X)$ for $i \geq 0$ and J. H. C. Whitehead’s theorem.

**Example 3.2.** Let $A_1, A_2, A_3, A_4$ be the subsets of the plane $\mathbb{R}^2$ given by

$A_1 = \{(x, \sin x^{-1}) \mid 0 < x \leq \frac{1}{\pi-1}\}$,

$A_2 = \{(\frac{1}{\pi-1}, y) \mid -2 \leq y \leq 0\}$,

$A_3 = \{(x, -2) \mid 0 \leq x \leq \frac{1}{\pi-1}\}$,

$A_4 = \{(0, y) \mid -2 \leq y \leq 1\}$.

Let $X = A_1 \cup A_2 \cup A_3 \cup A_4$. Let $x_0 = (0, 1)$ be chosen as the base point in $X$. It is known that the space $\Omega(X, x_0)$ is contractible. However $X$ is not contractible. In fact, the Čech homology $\tilde{H}_1(X) \simeq \mathbb{Z}$; whereas the singular homology group $H_1(X) = 0$. Hence, $X$ is not even of the homotopy type of a CW-complex.

**Remark 3.3.** Suppose $X$ is a 0-connected noncontractible space with $\Omega(X)$ contractible. From Lemma 3.1 we immediately get that such an $X$ will not be of the homotopy type of a CW-complex.

**Remark 3.4.** Let $X$ be the space $A_1 \cup A_2 \cup A_3 \cup A_4$ given in Example 3.2. Using the fact that Čech cohomology theory satisfies the axioms of Eilenberg-Steenrod we get, in the usual way as a consequence of the exactness homotopy and excision axioms, $\tilde{H}^{i+1}(\Sigma X) \simeq \tilde{H}^i(X)$ for $i \geq 1$. In particular, $\tilde{H}^2(\Sigma X) \simeq \tilde{H}^1(X) \simeq \mathbb{Z}$. Hence, $\Sigma X$ is not contractible. The same argument (repeated) yields that none of the spaces $\Sigma^l X$ ($l \geq 1$) is contractible.

It might be interesting to find an example of a topological space $X$ such that both $\Sigma X$ and $\Omega X$ are contractible but $X$ itself is not.
REFERENCES


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