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# SOME REMARKS ON WEIERSTRASS POINTS

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ABSTRACT. The author proves that, at a point $P$ on a closed Riemann surface of genus $g$, if $h$ is the first nongap at $P$ and $k$ is relatively prime to $h$, then $k$ is a gap if $g>\frac{1}{2}(h-1)(k-1)$. A consequence is that at the Weierstrass points of a closed Riemann surface, if the first nongap is a prime, the situation mirrors that in the hyperelliptic case, at least in a limiting sense.

1. In the study of closed Riemann surfaces an important topic is the question of the existence of meromorphic functions with prescribed poles. If we confine our attention to functions with poles at a single point $P$, the basic result is the Weierstrass gap theorem which says that for a surface $S$ of genus $g \geqq 1$ there are exactly $g$ integers $\beta_j, j=1, \cdots, g$,

$$1=\beta_1<\beta_2<\cdots<\beta_g \leqq 2g-1$$

such that there is no meromorphic function on $S$ whose only pole is one of order $\beta_j$ at $P$. For all but a finite number of points on $S$, $\beta_j=j, j=1, \cdots, g$. The exceptional points are called Weierstrass points. If $g=1$ there are evidently no exceptional points. The values $\beta_j$ are called the sequence of gaps, the complementary set with respect to the positive integers the sequence of nongaps (at $P$).

For a hyperelliptic surface ($g \geqq 2$) the sequence of gaps at a Weierstrass point is $1, 3, \cdots, 2g-1$. This corresponds to having 2 as the first nongap. Farkas [**1**] showed that for any surface if the first nongap is 3 and $g \geqq 4$ then 4 is a gap. His proof initially used Clifford's theorem, although he subsequently recognized that the result is an immediate consequence of the Weierstrass gap theorem. While the particular numbers 3 and 4 had some special significance in his context it is readily seen that they essentially have no really unique role in this situation. Further if the first nongap is a prime a situation similar to the hyperelliptic case exists at least in a limiting sense.

2. THEOREM 1. *Let $S$ be a closed Riemann surface of genus $g$, $P$ a point of $S$. If $h$ is the first nongap at $P$, and $k$ is relatively prime to $h$, then $k$ is a gap if $g>\frac{1}{2}(h-1)(k-1)$.*

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If $k$ were a nongap the same would be true for every positive integer of the form $hx+ky$ with $x$, $y$ nonnegative integers. It is well known that every integer not less than $(h-1)(k-1)$ has such a representation while exactly half of the integers $n$ satisfying

$$0 \leqq n < (h-1)(k-1)$$

have such a representation. (These statements follow at once from the considerations in [**2**], starting with p. 59.) There would then be at most $\frac{1}{2}(h-1)(k-1)$ gaps, and if $g>\frac{1}{2}(h-1)(k-1)$ this would contradict the Weierstrass gap theorem.

COROLLARY 1. *Let $S$ be a closed Riemann surface of genus $g$, $P$ a point of $S$. If the first nongap $h$ at $P$ is a prime then every nongap less than $2g(h-1)^{-1}+1$ is a multiple of $h$.*

This generalizes the situation in the hyperelliptic case.

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