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AUTOMORPHISMS AND TENSOR PRODUCTS OF ALGEBRAS

JOHN W. BUNCE[1]

ABSTRACT. In this note we prove that if A is a complex Banach algebra with identity, then the automorphism on $A\hat{\otimes}A$ determined by $\theta(a\otimes b)=b\otimes a$ is inner if and only if $A=M_n(C)$.

Let A be a C^*-algebra and let $A \otimes^* A$ be the C^*-tensor product of A with itself. Let θ be the automorphism of $A \otimes^* A$ determined by $\theta(a\otimes b)=b\otimes a$ for all a, b in A. S. Sakai proved in [3] that θ is inner if and only if A is the algebra $M_n(C)$ of all $n\times n$ complex matrices for some positive integer n. In an invited talk at the International Conference on Banach Spaces, Wabash College, June, 1973, Sakai asked if the theorem had an extension to general Banach algebras. In Theorem 1 of this note we prove that if A is a complex Banach algebra with identity of norm one, then the automorphism on $A\hat{\otimes}A$ determined by $\theta(a\otimes b)=b\otimes a$ is inner if and only if $A=M_n(C)$. In Theorem 2 we prove that if A is an algebra over an algebraically closed field F, then the automorphism θ as defined above on the algebraic tensor product $A\otimes_F A$ is inner if and only if $A=M_n(F)$. The proofs of these theorems are much easier than the proof of the C^*-algebra theorem.

Let A be a Banach algebra with identity e; we assume $\|e\|=1$. Let $A\otimes A$ be the algebraic tensor product of the complex vector space A with itself, and let $A\hat{\otimes}A$ be the completion of $A\otimes A$ in the greatest crossnorm [4]. The greatest crossnorm on $A\otimes A$ is an algebra norm [1], so $A\hat{\otimes}A$ is a Banach algebra. It is clear that the map θ from $A\hat{\otimes}A$ to $A\hat{\otimes}A$ determined by $\theta(a\otimes b)=b\otimes a$, for all a, b in A, is an automorphism. If $A=M_n(C)$, then $A\hat{\otimes}A$ is algebraically isomorphic to $M_{n^2}(C)$, and it is well known that every automorphism of $M_{n^2}(C)$ is inner.

THEOREM 1. *If the automorphism θ of $A\hat{\otimes}A$ is inner, then $A=M_n(C)$ for some positive integer n.*

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PROOF. We first prove that A is a simple algebra. Let I be a closed ideal in A with $I \neq A$. Let p be the quotient map from A to A/I and consider the canonical map [5, p. 439] $p\hat{\otimes}\mathrm{id}: A\hat{\otimes}A \to (A/I)\hat{\otimes}A$, where id is the identity map of A. Then $p\hat{\otimes}\mathrm{id}$ is clearly a homomorphism, and by [5, p. 445] $\mathrm{Ker}(p\hat{\otimes}\mathrm{id}) = \mathrm{cl}(I\otimes A)$, the closure of the space spanned by the elementary tensors $b\otimes a$, $b \in I$, $a \in A$. Since θ is inner we have that $\theta(\mathrm{cl}(I\otimes A)) \subseteq \mathrm{cl}(I\otimes A)$. If $a \in I$ we then have $\theta(a\otimes e) \in \mathrm{Ker}(p\hat{\otimes}\mathrm{id})$, so $(p\hat{\otimes}\mathrm{id})(e\otimes a) = (e+I)\otimes a = 0$, and $a=0$. Hence $I=\{0\}$ and A has no nontrivial closed ideals, and in fact no nontrivial ideals since A is complete and has an identity. By the classical Wedderburn-Artin theorem [2, Theorem 2.1.8], we now need only show that A is finite dimensional. Let $d \in A\hat{\otimes}A$ implement the inner automorphism θ, so that $b\otimes a = d(a\otimes b)d^{-1}$ for all $a, b \in A$. Now choose z and w in the algebraic tensor product $A\otimes A$ with the property that

$$\|z - d\| < (4\,\|d^{-1}\|)^{-1}, \qquad \|z\| < \|d\| + 1,$$
$$\|w - d^{-1}\| < (4(\|d\| + 1))^{-1}.$$

Let $z = \sum_{i=1}^{r} x_i\otimes y_i$, $w = \sum_{j=1}^{s} u_j\otimes v_j$. Then if a, b are in A we have

$$\begin{aligned}\|b \otimes a - z(a \otimes b)w\| \leqq{}& \|b \otimes a - d(a \otimes b)d^{-1}\| \\ &+ \|d(a \otimes b)d^{-1} - z(a \otimes b)d^{-1}\| \\ &+ \|z(a \otimes b)d^{-1} - z(a \otimes b)w\|\end{aligned}$$

which is less than or equal to $(\|a\|\,\|b\|)/2$. Thus for all a in A we have $\|e\otimes a - z(a\otimes e)w\| \leqq \|a\|/2$, and hence for all a in A,

$$(*) \qquad \left\|e \otimes a - \sum x_i a u_j \otimes y_i v_j\right\| \leqq \|a\|/2,$$

where the sum is over all $1 \leqq i \leqq r$, $1 \leqq j \leqq s$.

Now if $f \in A^*$, the bilinear function from $A\times A$ to A defined by $(a, b) \to f(a)b$ determines a linear function F from $A\hat{\otimes}A$ to A with the property that $\|F\| = \|f\|$ and $F(a\otimes b) = f(a)b$ for all a, b in A [5, Proposition 43.12]. Choose $f \in A^*$ such that $f(e) = 1 = \|f\|$ and apply the corresponding F to the equation (*) to obtain

$$\left\|a - \sum f(x_i a u_j) y_i v_j\right\| \leqq \|a\|/2,$$

for all $a \in A$. Now let H be the finite-dimensional space spanned by the set $\{y_i v_j : 1 \leqq i \leqq r, 1 \leqq j \leqq s\}$. We have shown that for all $a \in A$, $H \cap B(a, \|a\|/2) \neq \varnothing$, where $B(a, \|a\|/2)$ is the closed ball of radius $\|a\|/2$ and center a. But by Riesz's lemma [6, p. 84], this fact forces H to equal A. Thus A is finite dimensional. Q.E.D.

THEOREM 2. *If A is an algebra with identity e over an algebraically closed field F and the automorphism of the algebraic tensor product $A \otimes_F A$ determined by $\theta(a\otimes b)=b\otimes a$ is inner, then $A=M_n(F)$ for some positive integer n.*

PROOF. The proof that A is a simple algebra is almost the same as the proof that A is simple in Theorem 1; we omit the details. Let $d \in A\otimes A$ be such that $b\otimes a=d(a\otimes b)d^{-1}$ for all $a, b \in A$. Let $d=\sum_{i=1}^{n} x_i\otimes y_i$, where we assume that the set $\{x_i\}$ is linearly independent [4, Lemma 1.1]. We will show that A is the linear span of $\{y_j : 1 \leqq j \leqq n\}$. For a, b in A we have

$$(**) \qquad \left(\sum x_i \otimes y_i\right)a \otimes b = b \otimes a\left(\sum x_i \otimes y_i\right).$$

Now if for some $a \in A$ and index i, $ay_i \notin \operatorname{span}\{y_j\}$, choose g in the algebraic dual A' of A such that $g(ay_i)=1$, $g(y_j)=0$ for all j. Let $G: A\otimes A \to A$ be the linear function determined by the bilinear function $(c, b) \to g(b)c$ on $A \times A$, set $b=e$ in (**), and apply G to obtain

$$\sum g(y_j)x_j a = \sum g(ay_j)x_j.$$

Hence $\sum g(ay_j)x_j=0$ but $g(ay_i)\neq 0$, which contradicts our assumption that the $\{x_j\}$ were linearly independent. Thus $\operatorname{span}\{y_j\}$ is a left ideal, and a symmetrical argument shows that $\operatorname{span}\{y_j\}$ is a two-sided ideal. But A is simple, so $A=\operatorname{span}\{y_j\}$ and is finite dimensional. The Wedderburn-Artin theorem again implies that $A=M_n(F)$. Q.E.D.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KANSAS, LAWRENCE, KANSAS 66044