PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 1, May 1974

THE ERGODIC DECOMPOSITION OF CONSERVATIVE BAIRE MEASURES[1]

MICHAEL L. STURGEON

ABSTRACT. Certain topological conditions on a Markov transition function are shown sufficient for an integral representation of conservative invariant Baire measures. The analysis incorporates the Choquet-Bishop-de Leeuw extension of the Krein-Milman theorem.

1. **Introduction.** A Markov process is prescribed here as a quartet $(X, \mathscr{B}, p(x, B), m)$, the components respectively a given state space, σ-algebra of events, Markov transition function, and any σ-finite initial measure satisfying the condition that all m-null events are invariant or equivalently that on $\mathscr{B}: m(B)=0 \Rightarrow m\{x: p(x, B)>0\}=0$. The requirement on m is the obverse of the usual notion of a nonsingular transition function and will be called *presubinvariance*, a sine-qua-non for measures either invariant or conservative with respect to a given transition function.

The conservativeness of m, for our purposes, is best defined following S. R. Foguel [4], as

$$X = C(m) \Leftrightarrow \text{on } \mathscr{B}: \sum_1^\infty p^n(x, B) \begin{array}{ll} = \infty & \text{on } \tilde{B} \\ = 0 & \text{elsewhere} \end{array} \quad \text{ae}(m),$$

where $\tilde{B}$ is a minimal superset for B satisfying $p(x, \tilde{B})=1_{\tilde{B}}$ ae(m). When m is known to be conservative, $\tilde{B}$ can be replaced by the m-equivalent set $\hat{B}=B\cup\{x: 0<\sum_1^\infty p^n(x, B)\}$.

The invariance of m is defined using the operator $(\cdot)T$ on signed measures, whereby m is invariant (subinvariant) iff on $\mathscr{B}$:

$$mT(B) \equiv \int p(x, B)\, dm(x) = (\leqq)\ m(B).$$

Received by the editors May 8, 1973 and, in revised form, August, 20, 1973.

AMS (MOS) subject classifications (1970). Primary 28A65, 60J99.

Key words and phrases. Ergodic decomposition, integral representation, invariant measures, conservative measures, Choquet representation, Markov transition functions, recurrence, Baire measures.

[1] This research was supported by Grants ONR 2216-26 and NSF GP-8770 at the University of California, San Diego, and represents a portion of the author's doctoral dissertation written under the guidance of Professor Murray Rosenblatt.

For invariant conservative m, the σ-algebra of invariant events, $\Sigma(m)$, is composed of sets B satisfying $\tilde{T}\,1_B = 1_B$ ae(m), where $\tilde{T}(\cdot)$ is an operator on $L_\infty(m)$ defined by

$$\tilde{T}(f) \equiv (d/dm)[(f^+\,dm)T] - (d/dm)[(f^-\,dm)T],$$

where f in $L_\infty(m)$ is decomposed because $f\,dm$ need not be a signed measure. Such invariant sets are identical with those more typically defined by the requirement that $p(x, B) = 1_B$ ae(m), by [**3**, p. 77]. The operator $\tilde{T}$ will prove useful when characterizing the geometry of ergodic m.

From here on, X is at least a σ-compact, locally compact Hausdorff space, $\mathscr{B}$ its Baire sets, and m is a Baire measure. $(\mathscr{N}, \tau)$ denotes the collection of all Baire measures on $\mathscr{B}$ with the weak topology τ induced by the continuous functions with compact support, $C_c(X)$. A net $\{\nu_\alpha\}$ in $\mathscr{N}$ converges weakly to m in $\mathscr{N}$ iff for all f in $C_c(X)$: $\int_X f\,d\nu_\alpha \to^\alpha \int_X f\,dm$. Note that $(\mathscr{N}, \tau)$ is a subspace of the locally convex topological vector space of linear, but not necessarily continuous, functionals on $C_c(X)$.

X can be written as $\bigcup_1^\infty K_n$ and $\bigcup_1^\infty \mathscr{O}_n$, where the K_n and $\mathscr{O}_n$ are, respectively, compact Baire sets and open bounded Baire sets such that $\mathscr{O}_n \subset K_n \subset \mathscr{O}_{n+1} \subset K_{n+1}$. Subsequently, there is a useful realization of X as $\bigcup_1^\infty E_n$ where the E_n are open bounded Baire sets identified as $E_1 = \mathscr{O}_1$, $E_2 = \mathscr{O}_2$, and for $n \geqq 3$, $E_n = \mathscr{O}_n - K_{n-2}$. Any element of X lies in at most two of the E_n. Using only those E_n for which $m(E_n) > 0$, the function

$$u(x) \equiv \sum_n 1_{E_n}(x) \cdot [2^n m(E_n)]^{-1}$$

lies in $L_1(m) \cap L_\infty(m)$, and one can quickly show, for invariant m, that the conservativeness of m can be written as

$$X = C(m) \Leftrightarrow \text{for all } E_n \text{ such that } m(E_n) > 0\colon \sum_1^\infty p^k(x, E_n) = \infty \quad \text{ae}(m) \text{ on } E_n.$$

In the following the notation E_n will refer to this particular realization of X.

2. **An integral representation.** Let $\mathscr{C}$ denote the conservative invariant elements of $\mathscr{N}$. It is easily shown that $\mathscr{C}$ is a positive convex cone. By definition, a presubinvariant m in $\mathscr{N}$ is ergodic iff $\Sigma(m)$ is trivial. The ray, $\rho(m)$, generated by nonzero m in $\mathscr{C}$ is extreme by definition iff convex representations of any element of ρ always involve only other elements of ρ.

2.1. PROPOSITION. *A nonzero element ν of $\mathscr{C}$ is ergodic$\Leftrightarrow\rho(\nu)$ is extreme.*

PROOF. Let ν be ergodic with $\nu=a\mu_1+(1-a)\mu_2$, $a\in(0,1)$, and $\mu_1,\mu_2\in\mathscr{C}$. Then $\mu_1\ll\nu$, $\mu_2\ll\nu$, and μ_1 and μ_2 cannot be mutually singular so that $\mu_i=\lambda_i+w_i$, where $\lambda_1\perp\mu_2$ and $\lambda_2\perp\mu_1$ and $0\neq w_1\ll\mu_2$, $0\neq w_2\ll\mu_1$.

Denoting $d\mu_2/d\nu$ by f_2, we have $0\leqq f_2$ is $\mathscr{B}$-measurable and finite ae(ν) on bounded sets. We can decompose $X=\bigcup_1^\infty B_n$, the B_n disjoint bounded Baire sets, so that the functions g_k defined as $(f_2\wedge k)\cdot 1_{\bigcup_1^k B_n}$ satisfy $g_k\uparrow f_2$ on X and g_k is in $L_1(\nu)\cap L_\infty(\nu)$. The operator $(\cdot)T$ acting on $L_1(\nu)$ via $(g_k)T=(d/d\nu)[(g_k\,d\nu)T]$ is well extended, see [3, pp. 4–5], to act on $\mathscr{B}$-measurable positive functions by $(f_2)T\equiv\lim_k(g_k)T$. It follows from the invariance of ν that $f_2T=f_2$ ae(ν), or since the g_k are in $L_\infty(\nu)$ and since $\tilde{T}(\cdot)$ acting on $L_\infty(\nu)$ is precisely this extension of $(\cdot)T$, that $\tilde{T}f_2=f_2$ ae(ν). By [3, p. 76], ν is conservative with respect to the adjoint of $\tilde{T}$. So by [3, p. 21] f_2 is $\Sigma(\nu)$-measurable and there exists a constant c_2 such that $\mu_2=c_2\nu$. Similarly, $d\mu_1/d\nu=c_1$ ae(ν), and so μ_1 and μ_2 lie on $\rho(\nu)$. The converse is quickly shown by contradiction and the proposition is proven.

Two nonzero ergodic elements of $\mathscr{C}$ are either mutually singular or proportional. Referring to [6], if $\mathscr{C}$ were weakly closed in $\mathscr{N}$, we would have the notion of caps, $\mathscr{H}$, in $\mathscr{C}$ defined as $\mathscr{H}\equiv\{\lambda\in\mathscr{C}:M(\lambda)\leqq 1\}$, using a gauge functional M. The set $\mathscr{H}_1\equiv\{\lambda\in\mathscr{C}:M(\lambda)=1\}$ will be called the cap lid. From [5, p. 236], the extreme points, $\mathrm{Ex}(\mathscr{H})$, of a cap in $\mathscr{C}$ are precisely the vertex $\{0\}$ and the elements in the cap lid located on extreme rays of $\mathscr{C}$. So for any cap $\mathscr{H}$, the set $\mathscr{E}(\mathscr{H}_1)\equiv\mathrm{Ex}(\mathscr{H})-\{0\}$ is exactly representative of all nonzero singular ergodic measures in the cone $\mathscr{C}$.

2.2. THEOREM (INTEGRAL REPRESENTATION). *If the transition function $p(x,B)$ satisfies conditions*

(i) $\mathscr{C}\neq\{0\}$,

(ii) $T(\cdot):C_c(X)\to C_c(X)$, *where* $Tf(x)\equiv\int f(y)p(x,dy)$,

(iii) *for any open bounded Baire set $\mathscr{O}$ such that $\hat{\mathscr{O}}$ is unbounded, the set $\{x\in\mathscr{O}:\sum_1^\infty p^n(x,\mathscr{O})<\infty\}$ is open, then every nontrivial m in $\mathscr{C}$ lies in some cap lid $\mathscr{H}_{1_m}$ and there is a probability measure p_m concentrated on $\mathscr{E}(\mathscr{H}_{1_m})$ so that for all B in $\mathscr{B}$: $m(B)=\int_{\mathscr{E}(\mathscr{H}_{1_m})}\lambda(B)\,dp_m(\lambda)$.*

PROOF. By a theorem of P. Meyer, [5, p. 238] and [6, p. 95], there is a cap $\mathscr{H}_m$ such that $m\in\mathscr{H}_{1_m}$ whenever $\mathscr{C}$ is weakly closed in $\mathscr{N}$. Under (ii), $(\cdot)T$ acting on measures is an operator on $\mathscr{N}$. Given a net $\{\nu_\alpha\}$ in $\mathscr{C}$, $m=\tau$-limit ν_α satisfies for all f in $C_c(X)$: $\int f\,d(\nu_\alpha T)=\int f\,d\nu_\alpha=\int Tf\,d\nu_\alpha\to^\alpha \int Tf\,dm=\int f\,d(mT)\Rightarrow\nu_\alpha T=\nu_\alpha\to^\alpha mT\Rightarrow mT=m$. Thus weak limit points of $\mathscr{C}$ are invariant. An example to follow shows that $\mathscr{C}$ is generally not closed under (i) and (ii) in the sense that weak limit points may not be conservative. If $m(\hat{E}_n)<\infty$, $\hat{E}_n\in\Sigma(m)\Rightarrow\hat{E}_n\subset C(m)$. We therefore concern ourselves with those E_n for which $m(\hat{E}_n)=\infty$, denoting a typical such E_n

in the remainder of this argument by $\mathcal{O}$ and the set $\{x \in \hat{\mathcal{O}}: \sum_1^\infty p^n(x, \mathcal{O}) = \infty\}$ by $\hat{\mathcal{O}}_\infty$. By (iii), $\mathcal{O}-\hat{\mathcal{O}}_\infty$ is an open subset of $\mathcal{O}$ avoiding $\hat{\mathcal{O}}_\infty$. Suppose $m(\mathcal{O}-\hat{\mathcal{O}}_\infty)>0$. Then there is a compact Baire $K\subset \mathcal{O}-\hat{\mathcal{O}}_\infty$, $f\in C_c(X)$ with $0\leqq f\leqq 1$ and $f=1$ on K and 0 on $(\mathcal{O}-\hat{\mathcal{O}}_\infty)^c$ such that $0<m(K)\leqq \int f\,dm$, while for all α, $\int f\,d\nu_\alpha \leqq \nu_\alpha(\mathcal{O}-\hat{\mathcal{O}}_\infty)=0$, a contradiction. Therefore, with the supplementary "recurrence" condition (iii), m is conservative and $\mathscr{C}$ is weakly closed in $\mathscr{N}$. Applying the Choquet-Bishop-de Leeuw theorem as described by R. Phelps in [**6**, pp. 30–31], there is a probability measure p_m on $(\mathscr{H}_m, \mathscr{S})$, where $\mathscr{S}$ is the σ-algebra generated by the Baire sets of $\mathscr{H}_m$ and the set $\mathrm{Ex}(\mathscr{H}_m)$, so that $p_m(\mathscr{E}(\mathscr{H}_{1_m}))=1$ and p_m represents m in the sense that for any continuous affine function ϕ on $\mathscr{H}_m$: $\phi(m)=\int_{\mathscr{H}_m}\phi(\lambda)\,dp_m(\lambda)$. Let K be any compact Baire set. There is a sequence $\{g_n\}$ in $C_c(X)$ such that $g_n \downarrow 1_K$ and a corresponding sequence $\{\phi_n\}$ of continuous $\mathscr{S}$-measurable affine functions on $\mathscr{H}_m$ such that $\phi_n(m)\equiv \int_X g_n\,dm = \int_{\mathscr{H}_m} dp_m(\lambda)\int_X g_n\,d\lambda$. For some $\hat{n}$, $\phi_{\hat{n}}(m)<\infty$ and so for each λ and $q=0, 1, 2, \cdots$, $\phi_{\hat{n}+q}(\lambda)\to^q \lambda(K)$, so that

$$m(K)=\int_{\mathscr{E}(\mathscr{H}_{1_m})}\lambda(K)\,dp_m(\lambda).$$

Referring to [**1**, p. 5], the class of compact Baire sets in $\mathscr{B}$ is a π-system. It is easily shown that the collection $\{B\in\mathscr{B}: \lambda(B)$ is $\mathscr{S}$-measurable and $m(B)=\int_{\mathscr{E}(\mathscr{H}_{1_m})}\lambda(B)\,dp_m(\lambda)\}$ is a d-system and so equal to $\mathscr{B}$, yielding the theorem.

A more motivated argument towards the conservativeness of m is provided by defining the support of m as in [**9**, p. 308]. Then under condition (ii), $\hat{\mathcal{O}}$ is open so that $\mathrm{supp}(m)\cap\hat{\mathcal{O}}$ and $\hat{\mathcal{O}}_\infty$ are "similar" in that both sets are nonvoid and unbounded, both necessarily intersect m-positive open Baire subsets of $\hat{\mathcal{O}}$, and under (iii): $\mathrm{supp}(m)\cap\mathcal{O}\subset\hat{\mathcal{O}}_\infty\cap\mathcal{O}\Rightarrow m(\mathcal{O}-\hat{\mathcal{O}}_\infty)=0$.

We remark that (ii) implies that each individual set $\{x: p^n(x, \mathcal{O})>0\}$ is bounded, but does not in view of the following example imply that their union, or $\hat{\mathcal{O}}$ in particular, is bounded.

3. **The necessity of the recurrence hypothesis.** An example, suggested by M. Rosenblatt, shows given (i) and (ii) that a supplementary condition like (iii) is necessary. Let X be the real line R with the usual topology and let Z denote the integers. A continuous function $h(y)$ on R is defined as follows for $0<p<\frac{1}{2}<q<1$ and $p+q=1$: for $y\leqq 0$, $h(y)\equiv q$; for $y\geqq 0$, $h(y)$ is defined in piecewise fashion on the unit interval containing integers i and $i+1$ as q for $y=i, i+1$, as p for $i+1/(i+2)\leqq y\leqq i+1-1/(i+2)$, as $q(1-\lambda)+\lambda p$ for $y=i+\lambda/(i+2)$ or for $y=i+1-\lambda/(i+2)$ with $0<\lambda<1$. Graphically, in each positive integral interval, $h(y)$ is an inverted trapezoid.

Then for y in R and B in $\mathscr{B}$, $p(y, B) \equiv h(y)\, 1_B\, (y+1) + [1-h(y)]\, 1_B\, (y-1)$ is a Markov transition function. For fixed x in $[0, 1)$, denote $S_x \equiv \{i+x, i \in Z\}$. The transition function $p(i+x, A)$ on $(S_x, \mathscr{B} \cap S_x)$ corresponds to a random walk on S_x, and for $x \in (0, 1)$ a conservative invariant measure for the corresponding process in $(R, \mathscr{B})$ is given by

$$\nu_x(x-i) = (p/q)^{i-1}(1-h(x))/q, \qquad \nu_x(x) = 1,$$

$$\nu_x(x+i) = [h(x)h(x+1)\cdots h(x+i-1)] \div [1-h(x+1)]\cdots$$
$$[1-h(x+i-1)][1-h(x+i)] \quad \text{for } i = 1, 2, 3, \text{etc.}$$

So condition (i) is satisfied. As well, condition (ii) is fulfilled since for f in $C_c(X)$,

$$\int f(y)p(z, dy) = h(z)f(z+1) + [1-h(z)]f(z-1).$$

However, the net $\{\nu_x\}$ in $\mathscr{C}$ converges weakly as $x \to 0$ to the invariant measure $m = \{(q/p)^i\}_{i \in Z}$ which is not conservative since for $p \neq q$ there are no recurrent states and so no conservative presubinvariant measures on Z. So $\mathscr{C}$ is not, in general, weakly closed under (i) and (ii) and for example need not have a compact base.

4. **Finite measures.** In [7, p. 100], Lemma 1 states that the space of regular Borel probability measures on a compact X is weak-star compact. We could substitute Baire sets for Borel sets in the argument to obtain the weak-star compactness of the space $\mathscr{R}$ of Baire probabilities on X. Let $\mathscr{P}$ be the convex subspace of invariant (and so conservative) Baire probabilities. Condition (ii), which now reads $T: C(X) \to C(X)$, ensures that $\mathscr{P}$ is weakly closed in $\mathscr{R}$. $\mathscr{P}$ is subsequently weakly compact since the weak and weak-star topologies inherited by $\mathscr{R}$ have the same subbasis with sets of the form $\{\nu \in \mathscr{R} : |\int f\, d\nu - \int f\, d\nu_0| < \varepsilon\}$. Condition (ii) ensures $\mathscr{P} \neq \{0\}$, from [7, p. 101]. Another result in [**8**] says that $\mathrm{Ex}(\mathscr{P})$ consists precisely of ergodic elements of $\mathscr{P}$ since ergodicity there can be shown to be equivalent to ours. So with little effort we have:

4.1. PROPOSITION. *Let X be compact. Then the condition $T(\cdot)$: $C(X) \to C(X)$ suffices to ensure that the set of invariant Baire probabilities, $\mathscr{P}$, is nonvoid and weakly compact and that for any μ in $\mathscr{P}$ there exists a probability measure q_μ so that for all B in $\mathscr{B}$: $\mu(B) = \int_{\mathscr{E}(\mathscr{P})} \lambda(B)\, dq_\mu(\lambda)$, where $\mathscr{E}(\mathscr{P})$ is the set of ergodic elements of $\mathscr{P}$.*

5. **Transformation invariance.** The invariance of finite measures is typically defined in representation oriented papers with respect to one or more measurable, measure-preserving transformations on X. See [**2**] for a history of such efforts.

In [6, Chapter 10], there is an application of the Choquet-et al. theorem for transformation-invariant probabilities which is convenient for comparison. Let X be compact and $\mathscr{P}$ be the collection of Baire probabilities invariant with respect to $p(x, B)$. Form the cartesian product $Y \equiv X^{Z}$ and the associated product σ-algebra $\mathscr{A} \equiv \mathscr{B}^{Z}$ which in fact, [5, p. 23], is the σ-algebra of Baire sets of Y. Denote the shift transformation-invariant measures by $\mathscr{P}_{\pi}$. There is an embedding $\Gamma: \mathscr{P} \to \mathscr{P}_{\pi}$ via $\mu \to P_{\mu}$ where P_{μ} is the Carathéodory extension of the measure induced on any finite X^{F}, $F \equiv \{j_1, \cdots, j_f\}$ by

$$P_{\mu,F}\{y \in X^{F} : y_{j_i} \in B_i, j_i \in F\} = \text{prob}\{x_0 \in B_1, x_1 \in B_2, \cdots, x_{f-1} \in B_f\}.$$

Moreover, using prediction theory, as in [7, p. 97] it can be shown that Γ preserves ergodicity. If Γ were surjective, one would then quickly subsume existing representations for shift transformation-invariant finite measures. But this is not the case. The process $\{y_j\}$ associated with P_{μ} is Markovian and stationary. For a given X, one need only produce a non-Markovian stationary process to exhibit a shift-invariant P which is not the image under Γ of any μ in $\mathscr{P}$.

References

1. R. M. Blumenthal and R. K. Getoor, *Markov processes and potential theory*, Pure and Appl. Math., vol. 29, Academic Press, New York, 1968. MR **41** #9348.

2. R. H. Farrell, *Representation of invariant measures*, Illinois J. Math. **6** (1962), 447–467. MR **27** #265.

3. S. R. Foguel, *The ergodic theory of Markov processes*, Van Nostrand Math. Studies, no. 21, Van Nostrand-Reinhold, New York, 1969. MR **41** #6299.

4. ———, *Remarks on conservative Markov processes*, Israel J. Math. **6** (1968), 381–383. MR **39** #4939.

5. P. A. Meyer, *Probability and potentials*, Blaisdell, Waltham, Mass., 1966. MR **34** #5119.

6. R. R. Phelps, *Lectures on Choquet's theorem*, Van Nostrand, Princeton, N.J., 1966. MR **33** #1690.

7. M. Rosenblatt, *Markov processes. Structure and asymptotic behavior*, Springer-Verlag, Berlin and New York, 1971.

8. ———, *Transition probability operators*, Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), Vol. II: Contributions to Probability Theory, Part 2, Univ. California Press, Berkeley, Calif., 1967, pp. 473–483. MR **35** #3742.

9. H. Royden, *Real analysis*, Macmillan, New York, 1968.

ARTHUR D. LITTLE, INC., CAMBRIDGE, MASSACHUSETTS 02140

Current address: 45–573 Loihi Street, Kaneohe, Hawaii