SUMS OF QUOTIENTS OF ADDITIVE FUNCTIONS

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Abstract. Denote by \( \omega(n) \) and \( \Omega(n) \) the number of distinct prime factors of \( n \) and the total number of prime factors of \( n \), respectively. Given any positive integer \( a \), we prove that

\[
\sum_{2 \leq n \leq x} \frac{\Omega(n)}{\omega(n)} = x + x \sum_{i=1}^{a} a_i (\log \log x)^i + O(x/\log \log x)^{a+1}),
\]

where \( a_1 = \sum_p 1/p(p-1) \) and all the other \( a_i \)'s are computable constants. This improves a previous result of R. L. Duncan.

Denote by \( \omega(n) \) and \( \Omega(n) \) the number of distinct prime factors of \( n \) and the total number of prime factors of \( n \), respectively. R. L. Duncan [3] proved that

\[
\sum_{2 \leq n \leq x} \frac{\Omega(n)}{\omega(n)} = x + O(x/\log \log x).
\]

Duncan's result was based on the elementary estimate

(1) \[
\sum_{2 \leq n \leq x} \frac{1}{\omega(n)} = O(x/\log \log x).
\]

In a previous paper [1], we gave estimates of \( \sum_{n \leq x} 1/f(n) \) for a large class of additive functions \( f(n) \) (where \( \sum' \) denotes summation over those values of \( n \) for which \( f(n) \neq 0 \)), which in particular improved considerably the estimate (1). Such sums were further studied by De Koninck and Galambos [2].

In this paper, we prove the following:

Theorem. Let \( a \) be an arbitrary positive integer; then

\[
\sum_{n \leq x} \frac{\Omega(n)}{\omega(n)} = x + x \sum_{i=1}^{a} a_i (\log \log x)^i + O(x/\log \log x)^{a+1}),
\]

where \( a_1 = \sum_p 1/p(p-1) \) and all the other \( a_i \)'s are computable constants.
PROOF. Let \( t \) and \( u \) be real numbers satisfying \(|t| \leq 1\), \(|u| \leq 1\). Then, for \( \Re s > 1 \), we have

\[
\sum_{n=1}^{\infty} \frac{t^{\Omega(n)} u^{\omega(n)}}{n^s} = \prod_p \left( 1 + \frac{tu}{p^s} + \frac{t^2u}{p^{2s}} + \frac{t^3u}{p^{3s}} + \cdots \right)
\]

\[
= (\zeta(s))^{tu} \prod_p \left( 1 - \frac{1}{p^s} \right)^{tu} \prod_p \left( 1 + \frac{tu}{p^s} + \frac{t^2u}{p^{2s}} + \frac{t^3u}{p^{3s}} + \cdots \right)
\]

\[
= (\zeta(s))^{tu} H(t, u; s),
\]

say \( (\zeta(s)) \) denotes the Riemann zeta-function.\]

Using a theorem of A. Selberg [5], as we did previously in [1], we obtain that

\[
\sum_{n \geq x} t^{\Omega(n)} u^{\omega(n)} = \left( H(t, u; 1)/\Gamma(tu) \right) x \log^{t_u - 1} x + O(x \log^{t_u - 2} x),
\]

uniformly for \(|t| \leq 1\), \(|u| \leq 1\), which certainly implies that

\[
\sum_{n \geq x} t^{\Omega(n)} u^{\omega(n)} = \frac{H(t, u; 1)}{\Gamma(tu)} x \log^{t_u - 1} x + O(x/\log x)
\]

(2)

uniformly for \(|t| \leq 1\), \(|u| \leq 1\).

Now differentiating both sides of (2) with respect to \( t \) gives

\[
\sum_{n \geq x} \Omega(n) t^{\Omega(n) - 1} u^{\omega(n)} = \frac{x}{\log x} \left\{ \log^{t_u} x \frac{d}{dt} \left( \frac{H(t, u; 1)}{\Gamma(tu)} \right) + \frac{H(t, u; 1)}{\Gamma(tu)} \cdot \log^{t_u} x \cdot \log \log x \cdot u + O(1) \right\},
\]

which, by setting \( t = 1 \) and dividing both sides by \( u \), becomes

\[
\sum_{n \geq x} \Omega(n) u^{\omega(n) - 1} = (x/\log x) \{ G(u) \log^u x + F(u) \log^u x \cdot \log \log x + O(1/u) \}
\]

(3)

uniformly for \(|u| \leq 1\), where

\[
G(u) = \frac{1}{u} \frac{d}{dt} \left( \frac{H(t, u; 1)}{\Gamma(tu)} \right) \bigg|_{t=1}
\]

and

\[
F(u) = \frac{H(1, u; 1)}{\Gamma(u)}
\]
We now proceed to integrate both sides of (3) with respect to $u$ between $\varepsilon(x) = (\log x)^{-1/2}$ and 1 ($x \geq 3$). First we have

$$\int_{\varepsilon(x)}^{1} \left( \sum_{2 \leq n \leq x} \Omega(n)u^{\omega(n)-1} \right) du = \sum_{2 \leq n \leq x} \frac{\Omega(n)}{\omega(n)} \int_{\varepsilon(x)}^{1} u^{\omega(n)-1} du$$

$$= \sum_{n \leq x} \frac{\Omega(n)}{\omega(n)} - \sum_{n \leq x} \frac{\Omega(n)}{\omega(n)} (\varepsilon(x))^{\omega(n)}$$

$$= \sum_{n \leq x} \frac{\Omega(n)}{\omega(n)} + O\left( (\varepsilon(x)) \sum_{2 \leq n \leq x} \Omega(n) \right),$$

since $\omega(n) \geq 1$ for $n \geq 2$. It can be proved [4] in an elementary way that $\sum_{2 \leq n \leq x} \Omega(n) = O(x \log \log x)$. Therefore,

$$\int_{\varepsilon(x)}^{1} \left( \sum_{2 \leq n \leq x} \Omega(n)u^{\omega(n)-1} \right) du = \sum_{n \leq x} \frac{\Omega(n)}{\omega(n)} + O(x(\log \log x)(\log x)^{-1/2}).$$

(4)

On the other hand, as in [1], repeated integration by parts yields

$$\int_{\varepsilon(x)}^{1} G(u) \log^u x du$$

$$= \log x \left( \frac{G(1)}{\log \log x} - \frac{G'(1)}{(\log \log x)^2} + \frac{G''(1)}{(\log \log x)^3} - \cdots \right)$$

$$+ \frac{(-1)^{a-1}G^{(a-1)}(1)}{(\log \log x)^a} + O\left( \frac{1}{(\log \log x)^{a+1}} \right)$$

(5)

$$+ \frac{(-1)^{a+1}}{(\log \log x)^{a+1}} \int_{\varepsilon(x)}^{1} G^{(a+1)}(u) \log^u x du$$

Similarly we obtain

$$\log \log x \int_{\varepsilon(x)}^{1} F(u) \log^u x du$$

$$= \log x \left( \frac{F(1)}{\log \log x} - \frac{F'(1)}{(\log \log x)^2} + \cdots \right)$$

$$+ \frac{(-1)^{a}F^{(a)}(1)}{(\log \log x)^a} + O\left( \frac{1}{(\log \log x)^{a+1}} \right).$$

(6)
Finally, 
\[
\frac{x}{\log x} \int_0^1 \frac{du}{u} = O\left(\frac{x \log (x)}{\log x}\right) = O\left(\frac{x \log \log x}{\log x}\right)
\]
(7)
\[
= O\left(\frac{1}{(\log \log x)^{x+1}}\right)
\]

Putting together relations (3), (4), (5), (6) and (7), we have that

\[
\sum_{n \leq x} \frac{\Omega(n)}{\omega(n)} = x \left(F(1) + \frac{G(1) - F'(1)}{\log \log x} - \frac{G'(1) - F''(1)}{(\log \log x)^2} + \cdots \right)
\]
\[
+ (-1)^{x-1} \frac{G^{(x-1)}(1) - F^{(x)}(1)}{(\log \log x)^x} + O\left(\frac{1}{(\log \log x)^{x+1}}\right)
\]

A quite simple computation shows that $F(1)=1$ and that $G(1)-F'(1)=\sum_p 1/p(p-1)$, which proves our Theorem.

From the above reasoning it is clear that similar estimates of $\sum_{n \leq x} g(n)f(n)$ could be obtained for a larger class of additive functions $f$ and $g$ along the lines of our previous paper [1].

REFERENCES


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