ON DIRICHLET’S THEOREM AND INFINITE PRIMES

CARTER WAID

ABSTRACT. It is shown that Dirichlet’s theorem on primes in an arithmetic progression is equivalent to the statement that every unit of a certain quotient ring $\mathbb{Z}$ of the nonstandard integers is the image of an infinite prime. The ring $\mathbb{Z}$ is the completion of $\mathbb{Z}$ relative to the “natural” topology on $\mathbb{Z}$.

1. Notation. Throughout this note $N$ shall denote the natural numbers, $\mathbb{Z}$ the rational integers, and $P$ the positive primes. We shall follow the approach of Machover and Hirschfeld, [2], in our use of nonstandard analysis. Thus $U$ is to be a universal set containing $N$ and $^*U$ will be a comprehensive [6, p. 446] enlargement of $U$. The nonstandard natural numbers $^*N$ can be expressed as $^*N=N\cup N_\infty$ where $N_\infty$ is the set of infinite natural numbers. Similarly, $^*P=P\cup P_\infty$, $P_\infty$ the set of infinite primes.

2. Lemma. Let $a, b$ be coprime integers. A necessary and sufficient condition that the sequence $|a+bn|$ $(n \in N)$ contains infinitely many primes is that $|a+bn|$ be an infinite prime for some nonstandard natural number $n$.

PROOF. Clear.

3. Completions of $\mathbb{Z}$. In a series of papers [4], [5], [6], Robinson derives the results of this section in a more general setting.

Let $\mu=\bigcap n \cdot ^*\mathbb{Z}$ $(n \in N)$. The external ideal $\mu$ of $^*\mathbb{Z}$ is the monad of 0 for the “natural” topology on $\mathbb{Z}$. It can be characterized both as the set of all nonstandard integers divisible by every nonzero standard integer and as the $^*\mathbb{Z}$-ideal generated by numbers of the form $n!$ where $n$ is an infinite natural number. Clearly $\mathbb{Z} \cap \mu=0$ so that $\mathbb{Z}$ imbeds naturally in $^*\mathbb{Z}$. By results of Robinson [3, p. 109] on completions of metric spaces, $\mathbb{Z}$ is the completion of $\mathbb{Z}$ with respect to the “natural” topology and hence is the ring of $\nu$-adic integers [1]. Similarly, let $p$ be a standard prime and set $\mu_p=\bigcap p^n \cdot ^*\mathbb{Z}$ $(n \in N)$. Then $\mu_p$ is the monad of 0 for the usual $p$-adic

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topology on $\mathbb{Z}$ and can be characterized as either the set of nonstandard integers divisible by every finite power of $p$ or as the $\ast\mathbb{Z}$-ideal generated by numbers of the form $p^n$, $n$ an infinite natural number. Thus $\mathbb{Z}_p = \ast\mathbb{Z}/\mu_p$ is the ring of $p$-adic integers. It is not difficult to show that $\mu = \bigcap \mu_p$ ($p \in \mathbb{P}$), and then using the fact that $\ast\mathbb{U}$ is comprehensive, that $\mathbb{Z} \simeq \prod \mathbb{Z}_p$ ($p \in \mathbb{P}$).

4. The units of $\mathbb{Z}$. Robinson [5, p. 770] notes that the units of $\mathbb{Z}$ are the residue classes of nonstandard integers which have no standard prime factors. Using Dirichlet’s theorem we can sharpen this result and perhaps shed some light on infinite primes. If $x \in \ast\mathbb{Z}$ we shall let $\tilde{x}$ denote its residue class in $\mathbb{Z}$.

**Theorem.** The units of $\mathbb{Z}$ are precisely the residue classes $\tilde{p}$ where $p$ ranges over the infinite primes.

**Proof.** If $p \in \mathbb{P}_\infty$ there is an infinite natural number $n < p$, and hence $n!$ and $p$ are prime. Thus $\mu + p \cdot \ast\mathbb{Z} = \ast\mathbb{Z}$ and $\tilde{p}$ is a unit of $\mathbb{Z}$.

Conversely, if $\tilde{a}$ is a unit of $\mathbb{Z}$, $a \cdot \ast\mathbb{Z} + \mu = \ast\mathbb{Z}$, hence $a$ and $b$ are co-prime for some nonzero $b \in \mu$. We may assume $a$ and $b$ are positive and, using Dirichlet’s theorem, conclude that $a + bn = p$ is prime for some $n \in \ast\mathbb{N}$. Since $b$ is infinite, so is $p$, and clearly $\tilde{a} = \tilde{p}$.

This theorem has an interesting converse which points to a possible nonstandard “elementary” proof of Dirichlet’s theorem.

**Theorem.** Assume that the units of $\mathbb{Z}$ are the residues of infinite primes. Then Dirichlet’s theorem holds.

**Proof.** Let $a$ and $b$ be standard coprime integers and consider the sequence $\{a+bn\}$ ($n \in \mathbb{N}$). If $k$ is any standard natural number, there is an $n \in \mathbb{N}$ such that $a + bn$ is relatively prime to $k$! (choose $n$ to be the largest factor of $k$! that is prime to $a$). Consequently, if $k$ is an infinite natural number, there is an $n \in \ast\mathbb{N}$ such that $a + bn$ and $k$! are relatively prime. Then $a + bn$ has no standard prime factor and so (see remark at beginning of this section) $(a+bn)^{-1}$ is a unit in $\mathbb{Z}$.

We consider two cases:

(i) If $b > 0$, $(a+bn)^{-1}$ is a unit in $\mathbb{Z}$ and, by our assumption, $a + bn = p + d$ for some infinite prime $p$ and $d \in \mu$. Since $b$ is standard it divides $d$, and setting $d = bD$ we see that $a + b(n - D) = p$. Since $p$ is positive infinite, $n - D$ must be positive infinite.

(ii) If $b < 0$, $(-a - bn)^{-1}$ is a unit in $\mathbb{Z}$ and by an argument similar to the one above, $-a - b(n + D) = p$ where again $n + D \in \ast\mathbb{N}$. In either case we have $|a + bk| = p$ for some $k \in \ast\mathbb{N}$. Dirichlet’s theorem follows from the Lemma.
BIBLIOGRAPHY


