PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 1, May 1974

GRAPHS OF MEASURABLE FUNCTIONS

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ABSTRACT. Necessary and sufficient conditions are given for the measurability of a function in terms of its graph.

1. **Introduction.** Let X_1 be an arbitrary nonempty set and let $\mathscr{A}$ be a σ-algebra of subsets of X_1. X_2 is a complete separable metric space and $\mathscr{B}_2$ is the Borel field in X_2. The Borel field is the minimal σ-algebra over the open sets. It X_1 is a topological space, then $\mathscr{B}_1$ is the Borel field in X_1. If $f: X_1 \to X_2$, then $G(f) = \{(x, y) | x \in X_1, y = f(x)\}$ and f is $\mathscr{B}_1(\mathscr{A})$-measurable if and only if $f^{-1}(O) \in \mathscr{B}_1(\mathscr{A})$ for every O open in X_2. It is known that: (1) if $X_2 = R$ and if f is $\mathscr{A}$-measurable, then $G(f) \in \mathscr{A} \times \mathscr{B}_2$, the product σ-algebra [**2**, p. 143]; and (2) if X_1 is a complete separable metric space, then f is $\mathscr{B}_1$-measurable if and only if $G(f)$ is a Borel subset of $X_1 \times X_2$ (Propositions 1 and 6 in §3). This characterization of Borel measurability has been used to define measurability of set-valued maps [**1**]. The following theorem gives the corresponding result to (2) for Lebesgue measurable functions. For completeness, the propositions on Borel and analytic sets needed to prove the Theorem and to establish (2) above are included in §3 on analytic sets which follows the proof of the Theorem.

2. **Main result.** Let μ be a measure on $\mathscr{B}_1$ and let $\bar{\mu}$ be the completion of μ and $\mathscr{A}$ the completion of $\mathscr{B}_1$. Let $\mathscr{N}$ be all the sets of $\bar{\mu}$ measure zero.

THEOREM. *Let X_1 be a complete separable metric space. Then f is $\mathscr{A}$-measurable if and only if $G(f) \in \mathscr{A} \times \mathscr{B}_2$.*

PROOF. a. Let $G(f) \in \mathscr{A} \times \mathscr{B}_2$.

1. If $E \in \mathscr{A} \times \mathscr{B}_2$, then there are sets $E_1 \in \mathscr{B}_1 \times \mathscr{B}_2$, $E_2 \subset X_1 \times X_2$, and $E_3 \in \mathscr{N} \times \mathscr{B}_2$ (product σ-ring) such that $E = E_1 \cup E_2$ and $E_2 \subset E_3$, because if $\mathscr{S}$ is all such unions, then $\mathscr{S}$ contains all the measurable rectangles in $\mathscr{A} \times \mathscr{B}_2$ and $\mathscr{S}$ is a monotone class.

2. Each element in $\mathscr{B}_1 \times \mathscr{B}_2$ is a Borel subset of $X_1 \times X_2$.

3. Let O be open in X_2 and let $G(f) = E_1 \cup E_2$, $E_2 \subset E_3$. Now $E_3 \subset N \times B$, for some $N \in \mathscr{N}$ and some $B \in \mathscr{B}_2$, because if $\mathscr{T}$ is all subsets of $X_1 \times X_2$

Received by the editors May 14, 1973.

AMS (MOS) subject classifications (1970). Primary 28A20.

Key words and phrases. Measurable sets, measurable functions, Borel measurable, Lebesgue measurable, analytic sets.

which are contained in a measurable rectangle in $\mathcal{N}\times\mathcal{B}_2$, then $\mathcal{T}$ is a σ-ring containing all the measurable rectangles in $\mathcal{N}\times\mathcal{B}_2$. Let $A_i=(X_1\times O)\cap E_i$, $i=1, 2, 3$. If P is the projection of $X_1\times X_2$ into X_1, then $f^{-1}(O)=P(A_1)\cup P(A_2)$. Now $P(A_2)\subset P(A_3)\subset N$, so $P(A_2)\in\mathcal{A}$.

4. We show that $P(A_1)\in\mathcal{A}$, and it follows that f is $\mathcal{A}$-measurable. Now $P(A_2)$ is contained in some $C\in\mathcal{B}_1$ of $\bar{\mu}$ measure zero. Let $D=X_1-C$ and let $\bar{D}$ be the closure of D. Define $g:X_1\rightarrow X_2$ so that its graph is $(E_1\cap(D\times X_2))\cup(C\times\{y_0\})$, some fixed $y_0\in X_2$. Let g^1 be g restricted to $\bar{D}$. Let $A_4=(D\times O)\cap E_1$ and let $A_5=(\bar{D}\times O)\cap G(g^1)$. Since $\bar{D}$ is a complete separable metric space we have that $P(A_5)$ and $\bar{D}-P(A_5)=P((\bar{D}\times(X_2-O))\cap G(g^1))$ are analytic subsets of $\bar{D}$ (Proposition 3). Therefore $P(A_5)$ is a Borel subset of $\bar{D}$ (Proposition 5). So there is an $F\in\mathcal{B}_1$ so that $P(A_5)=F\cap\bar{D}$. Then $P(A_4)=F\cap D$ and $P(A_4)$ is a Borel subset of X_1. But $P(A_1)=P(A_4)\cup(P(A_1)\cap C)$ and $P(A_1)\cap C\in\mathcal{N}$, so that $P(A_1)\in\mathcal{A}$.

b. Let f be $\mathcal{A}$-measurable. For each n choose a disjoint sequence $\{A_{i,n}\}_{i=1}^{\infty}$ of elements in $\mathcal{B}_2$ whose union is X_2 and whose diameter is less than $1/n$. Then

$$G(f)=\bigcap_{n=1}^{\infty}\bigcup_{i=1}^{\infty}(f^{-1}(A_{i,n})\times A_{i,n})\in\mathcal{A}\times\mathcal{B}_2,$$

because $f^{-1}(\mathcal{B}_2)\subset\mathcal{A}$.

COROLLARY. *Let $f:R^n\rightarrow R$, let $\mathcal{A}$ be the Lebesgue subsets of R^n, and let $\mathcal{B}$ be the Borel subsets of R. Then f is Legesgue measurable if and only if its graph is measurable with respect to the product σ-algebra $\mathcal{A}\times\mathcal{B}$.*

3. **Analytic sets.** If T is a topological space, then the analytic subsets of T are the continuous images of the Borel subsets of T. X_1 and X_2 are complete separable metric spaces, I is the irrational numbers in $(0, 1)$ with the usual topology, N is the positive integers (discrete topology), and d_2 is the metric on X_2.

PROPOSITION 1. *If $f:X_1\rightarrow X_2$ is $\mathcal{B}_1$-measurable, then $G(f)$ is a Borel subset of $X_1\times X_2$* [**3**, *pp.* 384, 457].

PROOF. The function $h(x_1, x_2)=d_2(f(x_1), x_2)$ is a Borel measurable mapping of $X_1\times X_2$ into the reals. Hence $G(f)=h^{-1}(0)$ is Borel.

PROPOSITION 2. *Every Borel subset of X_1 is the continuous image of I* [**3**, *p.* 446].

PROOF. Let $\mathcal{S}$ be all subsets of X_1 which are continuous images of I. Since every complete separable metric space is the continuous image of I [**3**, p. 440], all the closed subsets of X_1 are in $\mathcal{S}$. Now $\mathcal{B}_1$ is the smallest class of subsets of X_1 which contains the closed sets and is closed under countable unions and intersections [**3**, p. 344]. It remains to show $\mathcal{S}$ is closed under countable unions and intersections. Let $A_i=f_i(I)\in\mathscr{S}$, $i=1,2,\cdots$, and let $A=\bigcup_{i=1}^{\infty}A_i$, $B=\bigcap_{i=1}^{\infty}A_i$. If I_n is the irrationals in $(n-1,n)$, $n\geqq 1$, define f on $I^*=\bigcup_{n=1}^{\infty}I_n$ as follows: $f(x)=f_n(x-n+1)$ if $x\in I_n$. Then f is continuous and $f(I^*)=A$. So, $A\in\mathscr{S}$ since I^* is homeomorphic to I.

Define $E=\{x=(x_1,x_2,\cdots)\in I^N|f_1(x_1)=f_2(x_2)=\cdots\}$. E is empty if and only if B is empty. Define f^* on E as follows: $f^*(x)=f_1(x_1)$. Then f^* is continuous and $f^*(E)=B$. Let $\bar{N}=N^N$. $\bar{N}$, with the usual metric, is a complete separable metric space. Since I and $\bar{N}$ are homeomorphic [**3**, p. 407], there is a homeomorphism $\lambda:\bar{N}^N\to I^N$. Also, $\bar{N}^N$ is a complete separable metric space. Now $\lambda^{-1}(E)$ is closed in $\bar{N}^N$ because E is closed in I^N and so $\lambda^{-1}(E)=f(I)$, f continuous. Hence $B=f^*\lambda f(I)$ and $B\in\mathscr{S}$.

PROPOSITION 3. *If $f:X_1\to X_2$ is continuous and $B\in\mathscr{B}_1$, then $f(B)$ is analytic* [**3**, *p.* 454].

PROOF. If X_2 is countable, then $f(B)$ is countable and in $\mathscr{B}_2$. Assume that X_2 is uncountable. Then there is a G_δ-set A in X_2 and a homeomorphism g between I and A [**3**, p. 445]. By Proposition 2, $B=h(I)$, h continuous. So $f(B)=fhg(A)$ is analytic.

PROPOSITION 4. *If A and B are two disjoint analytic subsets of X_1, then there is a Borel set $E\subset X_1$ so that $A\subset E$ and $E\cap B$ is empty* [**3**, *p.* 485].

PROPOSITION 5. *If A and A^c are analytic subsets of X_1, then A is a Borel set* [**3**, *p.* 486].

PROOF. Let $B=A^c$ in Proposition 4.

PROPOSITION 6. *If $f:X_1\to X_2$ such that $G(f)$ is a Borel subset of $X_1\times X_2$, then f is $\mathscr{B}_1$-measurable* [**3**, *p.* 489].

PROOF. Let O be open in X_2. By Proposition 3, $f^{-1}(O)$ is analytic in X_1 because it is the projection of Borel $G(f)\cap(X_1\times O)$ into X_1. Similarly, $(f^{-1}(O))^c=f^{-1}(O^c)$ is analytic in X_1. Therefore $f^{-1}(O)$ is Borel by Proposition 5.

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