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FIXED POINTS BY A NEW ITERATION METHOD

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ABSTRACT. The following result is shown. If T is a lipschitzian pseudo-contractive map of a compact convex subset E of a Hilbert space into itself and x_1 is any point in E, then a certain mean value sequence defined by $x_{n+1}=\alpha_n T[\beta_n Tx_n+(1-\beta_n)x_n]+(1-\alpha_n)x_n$ converges strongly to a fixed point of T, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers that satisfy some conditions.

It was recently shown in [1] that a mean value iteration method is available to find a fixed point of a strictly pseudo-contractive map. In this paper we shall prove that a certain sequence of points which is iteratively defined converges always to a fixed point of a lipschitzian pseudo-contractive map. For the definitions of a strictly pseudo-contractive map and a pseudo-contractive map in a Hilbert space, see, for example, [3].

THEOREM. *If E is a convex compact subset of a Hilbert space H, T is a lipschitzian pseudo-contractive map from E into itself and x_1 is any point in E, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T, where x_n is defined iteratively for each positive integer n by*

$$x_{n+1} = \alpha_n T[\beta_n Tx_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n, \tag{1}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences of positive numbers that satisfy the following three conditions:

$$0 \leqq \alpha_n \leqq \beta_n \leqq 1 \quad \textit{for all positive integers } n, \tag{2}$$

$$\lim_{n\to\infty} \beta_n = 0, \tag{3}$$

$$\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty. \tag{4}$$

As a particular case, we may choose for instance $\alpha_n=\beta_n=n^{-1/2}$.

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PROOF. We have, for any x, y, z in a Hilbert space H and a real number λ,

$$\begin{aligned}
\|\lambda x + (1-\lambda)y - z\|^2 &= \|\lambda(x-y) + y - z\|^2 \\
&= \lambda^2 \|x-y\|^2 + \|y-z\|^2 + 2\lambda \operatorname{Re}(x-y, y-z) \\
&= \lambda^2 \|x-y\|^2 + \|y-z\|^2 \\
&\quad + \lambda \operatorname{Re}[(\|x\|^2 - 2(x,z) + \|z\|^2) \\
&\qquad - (\|x\|^2 - 2(x,y) + \|y\|^2) - (\|z\|^2 - 2(z,y) + \|y\|^2)] \qquad (5)\\
&= \lambda^2 \|x-y\|^2 + \|y-z\|^2 + \lambda(\|x-z\|^2 - \|x-y\|^2 - \|y-z\|^2) \\
&= \lambda \|x-z\|^2 + (1-\lambda)\|y-z\|^2 - \lambda(1-\lambda)\|x-y\|^2.
\end{aligned}$$

Since T is pseudo-contractive, for any x, y in E,

$$\|Tx - Ty\|^2 \leqq \|x-y\|^2 + \|(I-T)x - (I-T)y\|^2, \qquad (6)$$

where I is an identity map.

From the assumption that T is lipschitzian, we also have that there is a positive number L such that

$$\|Tx - Ty\| \leqq L\|x-y\| \quad \text{for any } x, y \text{ in } E. \qquad (7)$$

From Schauder's fixed point theorem, $F(T)$, the set of fixed points of T, is nonempty since E is a convex compact set and T is continuous. Let p denote any point of $F(T)$.

We have, from (5) in which λ stands for α_n or β_n, the following three equalities, namely

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n T[\beta_n Tx_n + (1-\beta_n)x_n] + (1-\alpha_n)x_n - p\|^2 \\
&= \alpha_n \|T[\beta_n Tx_n + (1-\beta_n)x_n] - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \qquad (8)\\
&\quad - \alpha_n(1-\alpha_n)\|T[\beta_n Tx_n + (1-\beta_n)x_n] - x_n\|^2,
\end{aligned}$$

$$\begin{aligned}
\|\beta_n Tx_n + (1-\beta_n)x_n - p\|^2 &= \beta_n \|Tx_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 \\
&\quad - \beta_n(1-\beta_n)\|Tx_n - x_n\|^2,
\end{aligned} \qquad (9)$$

and

$$\begin{aligned}
&\|\beta_n Tx_n + (1-\beta_n)x_n - T[\beta_n Tx_n + (1-\beta_n)x_n]\|^2 \\
&\quad = \beta_n \|Tx_n - T[\beta_n Tx_n + (1-\beta_n)x_n]\|^2 \\
&\qquad + (1-\beta_n)\|x_n - T[\beta_n Tx_n + (1-\beta_n)x_n]\|^2 \qquad (10)\\
&\qquad - \beta_n(1-\beta_n)\|Tx_n - x_n\|^2.
\end{aligned}$$

Moreover from (6) we have the following two inequalities, namely

$$\begin{aligned}
\|T[\beta_n Tx_n + (1-\beta_n)x_n] - p\|^2 &= \|T[\beta_n Tx_n + (1-\beta_n)x_n] - Tp\|^2 \\
&\leqq \|\beta_n Tx_n + (1-\beta_n)x_n - p\|^2 \qquad (11)\\
&\quad + \|\beta_n Tx_n + (1-\beta_n)x_n - T[\beta_n Tx_n + (1-\beta_n)x_n]\|^2,
\end{aligned}$$

and

$$\|Tx_n - p\|^2 = \|Tx_n - Tp\|^2 \leqq \|x_n - p\|^2 + \|x_n - Tx_n\|^2. \tag{12}$$

Performing the calculation according to (8) $+\alpha_n[(9)+(10)+(11)+\beta_n \cdot (12)]$ side by side and eliminating common terms on both sides of the resulting inequality, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 \leqq \|x_n - p\|^2 &- \alpha_n\beta_n(1 - 2\beta_n)\,\|Tx_n - x_n\|^2 \\ &+ \alpha_n\beta_n\,\|Tx_n - T[\beta_n Tx_n + (1 - \beta_n)x_n]\|^2 \\ &- \alpha_n(\beta_n - \alpha_n)\,\|x_n - T[\beta_n Tx_n + (1 - \beta_n)x_n]\|^2,\end{aligned}$$

and it follows from condition (2) that

$$\begin{aligned}\|x_{n+1} - p\|^2 \leqq \|x_n - p\|^2 &- \alpha_n\beta_n(1 - 2\beta_n)\,\|Tx_n - x_n\|^2 \\ &+ \alpha_n\beta_n\,\|Tx_n - T[\beta_n Tx_n + (1 - \beta_n)x_n]\|^2.\end{aligned} \tag{13}$$

Now since T is lipschitzian, we have, from (7),

$$\|Tx_n - T[\beta_n Tx_n + (1 - \beta_n)x_n]\| \leqq L\beta_n\,\|Tx_n - x_n\|. \tag{14}$$

We have finally from (13) and (14),

$$\|x_{n+1} - p\|^2 \leqq \|x_n - p\|^2 - \alpha_n\beta_n(1 - 2\beta_n - L^2\beta_n^2)\,\|Tx_n - x_n\|^2. \tag{15}$$

Therefore adding these inequalities with $m, m+1, \cdots, n$ for n, we derive the following inequality

$$\|x_{n+1} - p\|^2 \leqq \|x_m - p\|^2 - \sum_{k=m}^{n} \alpha_k\beta_k(1 - 2\beta_k - L^2\beta_k^2)\,\|Tx_k - x_k\|^2,$$

from which we have

$$\sum_{k=m}^{n} \alpha_k\beta_k(1 - 2\beta_k - L^2\beta_k^2)\,\|Tx_k - x_k\|^2 \leqq \|x_m - p\|^2 - \|x_{n+1} - p\|^2.$$

From condition (3), there is some positive integer N such that $2\beta_k + L^2\beta_k^2 < \frac{1}{2}$ for all integers $k \geqq N$. Then if m is larger than N, we can get the following inequality

$$\frac{1}{2}\sum_{k=m}^{n} \alpha_k\beta_k\,\|Tx_k - x_k\|^2 \leqq \|Tx_m - p\|^2 - \|Tx_{n+1} - p\|^2.$$

The last member is bounded since E is a bounded set. Therefore the series on the left hand side is bounded. From condition (4) this should imply that $\liminf_{n\to\infty}\|Tx_n - x_n\| = 0$, which in turn implies from the compactness of E that there is a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ that converges to a certain point q of $F(T)$.

Since q is a fixed point of T, from (15) we see that if $n \geqq N$,

$$\|x_{n+1} - q\| \leqq \|x_n - q\|. \tag{16}$$

Let ε be any positive number. Then there is an n_{i0} such that $\|x_{n_{i0}}-q\| \leqq \varepsilon$ and $n_{i0} \geqq N$. Hence from (16), $\|x_n-q\| \leqq \varepsilon$ for $n \geqq n_{i0}$.

This completes the proof of the theorem.

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References

1. G. G. Johnson, *Fixed points by mean value iterations*, Proc. Amer. Math. Soc. **34** (1972), 193–194. MR **45** #1006.

2. W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510. MR **14**, 988.

3. F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces*, J. Math. Anal. Appl. **20** (1967), 197–228. MR **36** #747.

4. R. L. Franks and R. P. Marzec, *A theorem on mean value iterations*, Proc. Amer. Math. Soc. **30** (1971), 324–326. MR **43** #6375.

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