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TWO CHARACTERISTIC PROPERTIES OF THE SPHERE

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ABSTRACT. Let S be a closed, convex surface in E^3 with positive Gaussian curvature K and let K_{II} be the curvature of its second fundamental form. It is shown that S is a sphere if $K_{\mathrm{II}} \equiv cK$ for some constant c or if $K_{\mathrm{II}} \equiv \sqrt{K}$.

A regular surface S in euclidean three-space E^3, with positive Gaussian curvature K, possesses a positive-definite second fundamental form II, if appropriately oriented. Various natural questions arising in connection with this second Riemannian metric on S have been recently investigated (see the references in [**1**]). In particular, if S is closed and sufficiently smooth (an *ovaloid* in short), R. Schneider [**1**] has shown that the constancy of the curvature K_{II} of II implies that S is a sphere. In this note we give two global characterizations of the sphere of a nature similar to Schneider's. We shall not consider the question of possible local versions of these results.

THEOREM 1. *If on the ovaloid S we have identically $K_{\mathrm{II}} = cK$ for some constant c, then S is a sphere.*

PROOF. Note that c must be positive, since S is compact. If ∇_{II} denotes the first Beltrami operator with respect to II, and H the mean curvature of S, then

$$K_{\mathrm{II}} = H + Q - (8K^2)^{-1}\nabla_{\mathrm{II}}K$$

identically on S, where Q is a certain nonnegative function [**1**]. It follows that $K_{\mathrm{II}} \geqq K^{1/2} - (8K^2)^{-1}\nabla_{\mathrm{II}}K$. Let P be a point on S where K attains its minimum; there we have $\nabla_{\mathrm{II}}K = 0$. Now

$$K(P) = c^{-1}K_{\mathrm{II}}(P) \geqq c^{-1}\sqrt{(K(P))},$$

hence $\sqrt{K} \geqq \sqrt{(K(P))} \geqq c^{-1}$. On the other hand, denoting by dA and dA_{II} the area elements of S with respect to first and second fundamental forms, by the Gauss-Bonnet theorem,

$$(1) \qquad 4\pi = \int K\,dA = \int K_{\mathrm{II}}\,dA_{\mathrm{II}} = \int K_{\mathrm{II}}\sqrt{K}\,dA = \int cK\sqrt{K}\,dA,$$

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since the spherical-image mapping is a diffeomorphism; thus

$$\int K(1 - c\sqrt{K})\, dA = 0,$$

so that $K \equiv c^{-2}$ and S is a sphere.

THEOREM 2. *On an ovaloid S either $K_{\mathrm{II}} - \sqrt{K}$ changes sign or S is a sphere.*

PROOF. From (1) we obtain immediately

$$\int \sqrt{K}(K_{\mathrm{II}} - \sqrt{K})\, dA = 0.$$

Therefore, it suffices to show that $K_{\mathrm{II}} \equiv \sqrt{K}$ is true only if S is a sphere. We argue now indirectly. We consider the function $f = K^{-1/2}H - 1$; clearly $f \geqq 0$ with equality at a point if and only if that point is an umbilic. Assume that S is not a sphere and let P be a point on S where $f(P) = \max_S f > 0$. Introduce line-of-curvature coordinates near P. Then, with the notation $\mathrm{I} = Edu^2 + Gdv^2$ and $\mathrm{II} = Ldu^2 + Ndv^2$ near P, the integrability conditions for S assume the form

$$\frac{LN}{EG} = K = \frac{-1}{2\sqrt{(EG)}}\left[\left(\frac{E_v}{\sqrt{(EG)}}\right)_v + \left(\frac{G_u}{\sqrt{(EG)}}\right)_u\right], \tag{2}$$

and

$$L_v = HE_v, \quad N_u = HG_u, \tag{3}$$

and, by definition,

$$K_{\mathrm{II}} = -\frac{1}{2\sqrt{(LN)}}\left[\left(\frac{L_v}{\sqrt{(LN)}}\right)_v + \left(\frac{N_u}{\sqrt{(LN)}}\right)_u\right]. \tag{4}$$

Substituting (2) and (4) in $K_{\mathrm{II}} = \sqrt{K}$ we obtain

$$\left(\frac{E_v}{\sqrt{(EG)}} - \frac{L_v}{\sqrt{(LN)}}\right)_v + \left(\frac{G_u}{\sqrt{(EG)}} - \frac{N_u}{\sqrt{(LN)}}\right)_u = 0,$$

and using (3):

$$\left(\frac{E_v}{\sqrt{(EG)}} f\right)_v + \left(\frac{G_u}{\sqrt{(EG)}} f\right)_u = 0.$$

Hence,

$$2EGfK = E_v f_v + G_u f_u,$$

a contradiction, since the right-hand side of this equality vanishes at P and the left-hand side is positive.

Reference

1. R. Schneider, *Closed convex hypersurfaces with second fundamental form of constant curvature*, Proc. Amer. Math. Soc. **35** (1972), 230–233.

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