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PROOF OF AN INEQUALITY FOR TRIGONOMETRIC POLYNOMIALS

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ABSTRACT. Let a trigonometric polynomial have only real roots. An inequality is proved, relating the average value of the trigonometric polynomial between two consecutive roots to its maximal absolute value.

Consider the functional

$$f(T) = \max_{a,b} \int_a^b dx\, |T(x)| \Big/ (b-a) \max_{|x| \leqq \pi} |T(x)|,$$

where T is a trigonometric polynomial (in the following abbreviated tp) of degree $n \geqq 1$ with real coefficients and $2n$ real roots, and a and b are two of these ($a \neq b$).

Evidently, $f(T) \leqq 1$. I shall prove that $f(T) \leqq 2/\pi$. As a corollary we have Erdös' conjecture [1] that

$$\int_{-\pi}^{\pi} dx\, |T(x)| \leqq 4 \cdot \max_{|x| \leqq \pi} |T(x)|.$$

A maximizing tp T for f is easily shown to exist. We can assume $\max_{|x| \leqq \pi} |T(x)| = 1$. Let a and b be the maximizing roots (we can assume $T(x) \neq 0$ for $a < x < b$).

T may have multiple roots. Assume first that $a = b$. Consider a fundamental sequence of tps $\{T_j\}$ defining T, so that each T_j has maximal absolute value 1. Then $\max_{|x| \leqq \pi} |T_j'(x)| \leqq n$ (Bernshtein), and $\int_{a_j}^{b_j} dx\, |T_j(x)| \leqq n \cdot ((b_j - a_j)/2)^2$, so that $f(T_j) \to 0$ for $j \to \infty$, a contradiction. A multiple root for T with multiplicity m will have multiplicity $m-1$ as root for T'. Between two consecutive multiple roots for T, T' will have at least one root, but T' cannot have more than 1 root here since, otherwise, T' would have more than $2n$ roots. Therefore T', like T, has exactly $2n$ real roots.

To prove $f(T) = 2/\pi$ we need only prove that a real constant c exists, for which $T(x) = \sin n(x-c)$.

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The general method of proof consists in showing that if T has another form, it is possible to choose a tp t of degree at most n, so that $f(T+\varepsilon t)>f(T)$; ε is a small positive real number. t must, of course, be chosen so that $T+\varepsilon t$ still has $2n$ real roots.

We consider changes of the first order in ε. The change $\varepsilon\delta_n$ of the numerator of f is determined from

$$\delta_n = \int_a^b dx\, t(x). \tag{1}$$

The change $\varepsilon\delta_d$ of the denominator of f is determined from

$$\delta_d = (b-a)\cdot \max_j \{t(z_j)\operatorname{sign}(T(z_j))\} + \delta_b - \delta_a, \tag{2}$$

where the sign of T is chosen so that $T(x)>0$ for $a<x<b$, and z_j are the points in which $|T(x)|$ assumes its maximal value 1. Moreover, $\delta_b=0$ for $t(b)=0$, and $\delta_b=-t(b)/T'(b)$ for $t(b)\neq 0$ (we shall have $t(b)\neq 0$ only if $T'(b)\neq 0$); similarly for a.

The inequality $f(T+\varepsilon t)>f(T)$ is, for sufficiently small ε, equivalent to

$$\delta_n > \delta_d \cdot f(T). \tag{3}$$

In the following we mostly specify t by indicating its root set or, rather, describing how this set differs from the root set of T. Note that t, apart from a multiplicative constant, is uniquely determined from its root set $\{y_j\}$ by $t(x)=\prod_{j=1}^{2n} \sin((x-y_j)/2)$. For simplicity we shall sometimes refer to a nonroot as a root of multiplicity zero. A root interval in which $|T(x)|$ does not assume its maximal value will be termed a regular interval.

(A) Consider first the case where (a, b) is regular.

(A1) $T(x)$ has a root y of multiplicity $m_y>1$. If y equals a or b, $y=a$ (say), we let t have the same roots as T, but a with multiplicity m_a-1 and b with multiplicity m_b+1. We require $t(x)>0$ for $a<x<b$. Then, according to (1), $\delta_n>0$; according to (2), $\delta_d<0$.

If $y\neq a, b$, we let t have the same roots as T, except for a, b, and y, which now shall be roots of multiplicity m_a+1, m_b+1, and m_y-2, respectively. As before, we require $t(x)>0$ for $a<x<b$. Then $\delta_n>0$ and $\delta_d<0$.

(A2) $T(x)$ has no multiple roots. Assume that T has another regular interval (x_1, x_2). If $x_2=a$, t and T should have the same roots except for x_1, which is not a root of t, and b, a double root of t. Similarly for $x_1=b$. Otherwise, t and T should have the same roots except for x_1 and x_2, which are nonroots of t, and a and b, which are double roots of t.

We are left with the case in which (a, b) is the only regular interval. Consider the tp $U(x)=T(\frac{1}{2}(a+b)+x)-T(\frac{1}{2}(a+b)-x)$. We have $U(x)=-U(-x)$ for all x, in particular $U(0)=U(\pi)=0$. Besides $U(\pm(b-a)/2)=0$.

In the points where $|T(\frac{1}{2}(a+b)+x)|=1$, $U(x)\cdot T(\frac{1}{2}(a+b)+x)\geqq 0$. We can assume that U has a zero between $(b-a)/2$ and the first such point. Let now x increase starting from $x=(b-a)/2$ and, pairing each of the points described above with the monotony interval to its left, we see that U must have at least $2n+1$ roots and therefore vanish identically. It is easily seen that the possibility of U having double roots does not invalidate this result.

We may set $(a+b)/2=0$ and write $T(x)=\cos nv$ with $\cos v=\cos^2 u\cdot\cos x-\sin^2 u$ and $0<u<\pi/4n$. We have

$$f(T)=\frac{1}{b}\int_0^b dx\cdot\cos nv.$$

The connection between b and u is

$$b=\arccos\frac{\sin^2 u+\cos(\pi/2n)}{\cos^2 u}=2\cdot\arccos\left(\frac{\cos(\pi/4n)}{\cos u}\right).$$

After some manipulation,

$$f(T)=\frac{2n}{\arccos(\cos(\pi/4n)/\cos u)}\int_u^{\pi/4n}dv\,\sin 2nv\cdot\arccos\left(\frac{\cos v}{\cos u}\right).$$

Now, the ratio $\arccos(\cos v/\cos u)/\arccos(\cos(\pi/4n)/\cos u)$ is easily shown to increase with decreasing u for fixed v. Then $f(T)$ increases with decreasing u, a contradiction.

(B) We now know that $|T(x)|$ assumes its maximal value in (a, b).

(B1) Assume first that T has two different multiple roots, x_1 and x_2. The roots of t shall be the same as those of T, except for a, whose multiplicity is increased by 1, if a is a simple root, and decreased by 1, if a equals x_1 or x_2, and similarly for b; furthermore the root $z\in(a, b)$ of T' shall be a double root for t, and if x_1 or x_2 differs from a and b, its multiplicity as root for t is decreased by 2; $t(x)\geqq 0$ for $a\leqq x\leqq b$.

Similarly, if T has a multiple root x_1 and a regular interval (x_2, x_3) for which x_1 is not an endpoint, we choose the roots of t as above, except for x_2 and x_3, whose multiplicities are reduced by 1. If $b=x_2$ (resp. $a=x_3$) the multiplicity of b (resp. a) is unchanged, otherwise it is increased by 1.

(B2) Let T have exactly one multiple root, which we can put equal to 0. If $a=0$ we put $t(x)=T'(x)$. If $b=0$ we put $t(x)=-T'(x)$. In both cases $\delta_n=0$ and $\delta_d=-1$, so that (3) is satisfied. Thus, neither a nor b is a multiple root of T.

We know that if a root interval (x_1, x_2) exists, in which $|T(x)|$ does not assume its maximal value, we must have either $x_1=0$ or $x_2=0$. Replacing

$T(x)$ by $T(-x)$, if necessary, we can assume that $x_1=0$. The least positive root of T' is denoted by r.

Putting $t(x)=-T'(x)\cdot\cot(x/2)$ we get

$$\delta_n = -\int_a^b dx\,\frac{T(x)}{1-\cos x}\,; \qquad \delta_d = \cot(b/2) - \cot(a/2).$$

The inequality (3) becomes

$$\int_a^b dx\, T(x)g(x) < 0, \tag{4}$$

where $g(x)=h(x)-(b-a)^{-1}\int_a^b dy\, h(y)$ and $h(x)=(1-\cos x)^{-1}$. As shown below, it suffices to prove

$$(T(x_1) = T(x_2) \wedge |x_1| < |x_2|) \Rightarrow |T'(x_1)| \leqq |T'(x_2)|, \tag{5}$$

where x_1 and x_2 belong to the same root interval $[a, b]$ for T, and $|x|$ denotes the distance from x to the set $\{2p\pi; p \text{ integral}\}$. Since in (5) only absolute values of abscissas are involved, and since $g(x)$ is an even function, we can restrict ourselves to the case $0<(a+b)/2\leqq\pi$, implying $|a|\leqq|b|$. As before, we assume $T(x)>0$ for $a<x<b$.

Assume that (5) is true. T' has exactly one root (call it z) in (a, b). Assume first that $\pi<b$. If then $z>\pi$ also, we obtain by integration from $(x, T)=(z, 1)$ that

$$(T(x_1) = T(x_2) \wedge x_1 < x_2) \Rightarrow |x_1| > |x_2|,$$

contradicting $|a|\leqq|b|$. Therefore, $z\leqq\pi$. Integrating from $T=0$ we get

$$(T(x_1) = T(x_2) \wedge x_1 < x_2) \Rightarrow x_1 - a \geqq b - x_2 \Rightarrow z \geqq (a+b)/2.$$

We shall prove that (5) implies (4). Assume, first, $b\leqq\pi$. Then $h(x)$ is strictly decreasing in (a, b), and $g(x)$ has exactly one root y in (a, b). Denoting by G a primitive function to g (4) is equivalent to

$$\int_{x=a}^{x=y} dG\; T(x_1(G)) < \int_{x=b}^{x=y} dG\; T(x_2(G)),$$

where $x_1(G)$ and $x_2(G)$ are the two inverse functions to $G(x)$. It suffices to prove $T(x_1(G))\leqq T(x_2(G))$ for all values of G between $G(a)=G(b)$ and $G(y)$, with strict inequality for at least one value of G. In fact, if we can show

$$x_1(G) + x_2(G) \leqq a + b \tag{6}$$

(strict inequality for at least one G), evidently (5) implies $T(x_1(G))\leqq T(x_2(G))$; either $x_2(G)\leqq z$, and the inequality is true, or $x_2(G)>z\geqq(a+b)/2$, and $T(x_1)\leqq T(a+b-x_1)\leqq T(x_2)$, as postulated.

According to definition, $G(x_1(G))=G(x_2(G))$, i.e.

$$\int_{x_1}^{x_2} dx\, g(x) = 0 \quad \text{or} \quad \frac{\int_{x_1}^{x_2} dx\, h(x)}{x_2 - x_1} = \frac{\int_a^b dx\, h(x)}{b - a} \tag{7}$$

The function

$$h_1(x_1) = \int_{x_1}^{x_2} dx\, h(x) \Big/ (x_2 - x_1)$$

has the derivative

$$h_1'(x_1) = (h_1(x_1) - h(x_1))/(x_2 - x_1);$$

as $h_1(x_1)$ is the average of $h(x)$ over (x_1, x_2), it is strictly decreasing, since $h(x)$ is.

Therefore, if (6) is not satisfied,

$$h_1(x_1) \leqq h_1(a+b-x_2). \tag{8}$$

The function

$$h_2(x) = \frac{1}{2x} \int_{(a+b)/2-x}^{(a+b)/2+x} dt\, h(t) \tag{9}$$

has the derivative

$$h_2'(x) = \frac{1}{2x^2}\left(x \cdot (h(\tfrac{1}{2}(a+b)+x) + h(\tfrac{1}{2}(a+b)-x)) - \int_{(a+b)/2-x}^{(a+b)/2+x} dt\, h(t)\right);$$

the numerator vanishes for $x=0$ and has the derivative

$$x(h'(\tfrac{1}{2}(a+b)+x) - h'(\tfrac{1}{2}(a+b)-x)),$$

which is positive, since $h''(x)=(2+\cos x)/(1-\cos x)^2$ is positive.

Therefore $h_2(x)$ is strictly increasing, and we have, according to (8) and (9),

$$\begin{aligned} h_1(x_1) &\leqq h_1(a+b-x_2) \\ &= h_2(x_2 - \tfrac{1}{2}(a+b)) < h_2(\tfrac{1}{2}(b-a)) \quad \text{for } x_2 < b. \end{aligned}$$

This contradicts (7), and (6) must be true.

Next, assume $b>\pi$. The function $h(x)$ is increasing for $x>\pi$, and $g(x)$ may have two roots in (a, b). Assume first that $g(x)$ has only one root in this interval. Then we can use the proof above with the complication that $h_1(x_1)$ is not decreasing for $x_2>\pi$ and x_1 close to π. But $x_1\leqq\pi$, and $h_1(\pi)$ is a local maximum for $h_1(x_1)$. In fact, $h_1'(x_1)=0\Rightarrow h_3(x_1)\equiv\int_{x_1}^{x_2} dx\,(h(x)-h(x_1))=0$; but $h_3'(x_1)=-(x_2-x_1)h'(x_1)\geqq 0$ for $x_1\leqq\pi$.

Therefore h_1' has one and only one root, $x_3<\pi$. We have $2\pi-x_2<x_3<\pi$, and always $a+b-x_2<x_3$; but if $x_1>x_3$ we have

$$h_1(x_1) \leqq h_1(\pi) = h_1(2\pi - x_2) < h_1(a + b - x_2),$$

which is all we need.

Next assume that $g(x)$ has two roots y_1 and y_2. Then at some point y_0 between a and b we must have $G(a)=G(y_0)$, and the inequality (4) can be written

$$\left(\int_{x=a}^{x=y_1} dG\ T(x_1(G)) - \int_{x=y_0}^{x=y_1} dG\ T(x_2(G))\right) + \left(-\int_{x=y_2}^{x=y_0} dG\ T(x_3(G)) + \int_{x=y_2}^{x=b} dG\ T(x_4(G))\right) < 0,$$

where $x_j(G)$ $(1\leqq j\leqq 4)$ are the 4 relevant inverse functions to $G(x)$. Following the method above we easily prove $T(x_1(G))\leqq T(x_2(G))$ (with strict inequality at least for one G); the proof will be finished, if we can prove $T(x_3(G))\geqq T(x_4(G))$. It suffices to prove $y_0\geqq\pi$, since $y_0\leqq x_3(G)\leqq x_4(G)$ and $z\leqq\pi$. But $\int_a^{y_0} dx\, g(x)=\int_a^b dx\, g(x)=0$, i.e.

$$\frac{1}{b - y_0}\int_{y_0}^{b} dx h(x) = \frac{1}{b - a}\int_a^b dx\ h(x).$$

If $y_0<\pi$ we would have $h_1(y_0)<h_1(a)$, a contradiction.

Finally we must show that condition (5) is fulfilled. I shall use the notation: τ is the set of tps of degree n and maximal absolute value 1, with real coefficients and $2n$ real roots. $T_{m,p}$ $(1\leqq m\leqq 2n$ and $0<p\leqq 1)$ is the (apart from the sign) unique tp $\in\tau$ satisfying:

(1) 0 is a root of exact multiplicity m.

(2) Denoting by r the least positive root of $T'_{m,p}$ we have $0<|T_{m,p}(r)|=p\leqq 1$.

(3) $|T_{m,p}(x)|=1$, if x equals one of the remaining $2n-m$ roots of $T'_{m,p}$.

The main tools will be two transformations $k_{1,m,p}$ and $k_{2,m,p}$ of one such tp into another. In fact, $k_{1,m,p}T_{m,1}=T_{m,p}$ and $k_{2,m,p}T_{m-1,1}=T_{m,p}$.

First $k_{1,m,p}$ will be investigated. In the next paragraphs we shall for notational convenience replace $T_{m,p}$ by T. We still let r denote the smallest positive root of T', and we have $|T(r)|=p$. Let $t(x)=T'(x)\sin\frac{1}{2}x/\sin\frac{1}{2}(x-r)$ and, for $x\neq r$, $\delta(x)=-\varepsilon\sin\frac{1}{2}x/\sin\frac{1}{2}(x-r)$. Then for $x\neq r$ we have

$$T(x + \delta(x)) + \varepsilon t(x + \delta(x)) = T(x) + O(\varepsilon^2).$$

The changes in the extremal values (other than $T(r)$) of T will be of

second order in ε, while the changes in their positions in general will be of first order in ε.

The change $\varepsilon t(r)$ in $T(r)$ will be of first order in ε, and $|T(r)|$ will decrease. Replacing $T(x)$ by $T(x)+\varepsilon t(x)$ we can define a new function $t(x)$ and transform again, etc.

It is seen that $k_{1,m,p}$ can be conceived as an infinite product of infinitesimal transformations of the type described above (for $\varepsilon\to 0$), operating on $T_{m,1}$ and producing $T_{m,p}$ as end result. Similarly, $k_{2,m,p}$ is generated partly by transformations corresponding to

$$t(x) = T'(x)\sin\frac{x}{2}\left(\frac{1}{\sin\frac{1}{2}(x-r)} - \frac{1}{\sin\frac{1}{2}(x+r)}\right) = T'(x)\frac{2\sin x\sin r/2}{\cos r - \cos x}$$

($T_{m-1,1}$ is hereby transformed into a tp $T_{m-1,p,p}\in\tau$ with 0 a root of multiplicity $m-1$ and $|T_{m-1,p,p}(r)|=p$, where r is the smallest positive root of $T'_{m-1,p,p}$; besides, $|T_{m-1,p,p}(x)|=|T_{m-1,p,p}(-x)|$ for $0\leqq x\leqq\pi$), partly by transformations corresponding to $t(x)=-T'(x)\sin\frac{1}{2}x/\sin\frac{1}{2}(x+r)$, whereby $T_{m-1,p,p}$ is transformed into $T_{m,p}$.

We shall prove (5) for every root interval (a, b) and every $m\geqq 1$. Let $a\leqq x\leqq b$. Replacing T by $T+\varepsilon t$ will, as noted above, apart from quantities of second order in ε, move a point $(x, T(x))$ to the point $(x+\varepsilon\delta(x), T(x))$, where (for $x\neq r$) $\delta(x)=-t(x)/T'(x)$ (or its limit). T' will be divided by $1+\varepsilon\delta'(x)$.

Consider, first, the case $t(x)=T'(x)\sin\frac{1}{2}x/\sin\frac{1}{2}(x-r)$, corresponding to $k_{1,m,p}$; $\delta'(x)=\sin\frac{1}{2}r/(1-\cos(x-r))$. Then $|x_1-r|<|x_2-r|\Rightarrow\delta'(x_1)>\delta'(x_2)$. For $m=1$ we can only use a k_1-transformation. But this suffices, since $T_{1,p}$ is an even function of $x-r$: For the root intervals containing r and $r-\pi$ (5) is satisfied with equality. Moreover, the transformation $k_{1,1,p}$ makes the left endpoint of the interval containing $\pi+r$ move from π towards the left, and for $p\to 0$, $z\to\pi$. But then for x_1 and x_2 belonging to one of the other root intervals we have $|x_1|<|x_2|\Rightarrow|x_1-r|<|x_2-r|$, and (5) is satisfied.

This was the first part of a proof of (5) by induction on m.

Let $m>1$. Assume that (5) is satisfied for $T_{k,p}$ with $1\leqq k\leqq m-1$ and $0<p\leqq 1$ (note that $\lim_{p\to 0} T_{m-1,p}=T_{m,1}$, if signs are chosen suitably). Assume that there is a smallest value $p_0>0$ for p, for which (5) is valid for $T_{m,p}$. Now we need some information concerning the transformations generating $k_{2,m,p}$. Corresponding to $t(x)=-T'(x)\sin\frac{1}{2}x/\sin\frac{1}{2}(x+r)$ we find $\delta'(x)=\sin\frac{1}{2}r/(1-\cos(x+r))$ so that $|x_1+r|<|x_2+r|\Rightarrow\delta'(x_1)>\delta'(x_2)$. Corresponding to

$$t(x) = T'(x)\sin\frac{x}{2}\left(\frac{1}{\sin\frac{1}{2}(x-r)} - \frac{1}{\sin\frac{1}{2}(x+r)}\right)$$

we find

$$\delta'(x) = \sin\frac{r}{2}\left(\frac{1}{1-\cos(x-r)} + \frac{1}{1-\cos(x+r)}\right),$$

which is an even function of x and a decreasing function of $|x|$ for $|x|>r$.

There are now two subcases.

(1) *m even.* $T'_{m,1}(\pi)=0$, and $k_{1,m,p}$ moves all points to the left, so that $\lim_{p\to 0} T_{m,p}(\pi)=0$. In particular, for all root intervals (a, b) with $0<z\leqq\pi$, (5) is fulfilled for $p<p_0$ too. For intervals with $-\pi<z<0$, $k_{2,m,p}$ is seen to work, so that (5) is fulfilled here too, and we obtain a contradiction.

(2) *m odd.* Here $k_{1,m,p}$ works as before, if $0<z<\pi$, and $k_{2,m,p}$ takes care of the remaining intervals. The interval containing $-\pi$ may need a more elaborated argument. In fact, after the first part of $k_{2,m,p}$, $T'_{m-1,p,p}(-\pi)=0$; during the last part of $k_{2,m,p}$ all points are moved to the right, so that for x_1, x_2 belonging to the interval in question $|x_1|<|x_2|\Rightarrow|x_1+r|<|x_2+r|\Rightarrow\delta'(x_1)>\delta'(x_2)$, so that (5) is fulfilled also for $p<p_0$.

(B3) We now know that T has only simple roots. Assume that T has a regular interval. T' has a root in this interval, which we set equal to zero. Putting $t(x)=-T'(x)\cot(x/2)$ we get the conditions (4) and (5) as before. If T has only one regular interval, we use the transformation generated by $t(x)=T'(x)\cot(x/2)$ to prove that (5) is satisfied. Next we consider the case where T has exactly two regular intervals, and we assume first that these intervals have one endpoint in common. One of the two corresponding roots for T' is zero, and we can assume that the other one (r, say) is positive. Then we first use the transformation described above to produce the right value for $x=0$, next we use the transformation generated by $t(x)=T'(x)\sin\frac{1}{2}x/\sin\frac{1}{2}(x-r)$ to show that (5) is fulfilled for $0<z<\pi$. During the first step the distance between π and the next extremum to its right becomes greater than the smallest positive root of T'. During the last step we find for the movement of the smallest positive root of T',

$$\delta(r) = -\left(\cos\frac{r}{2} + \frac{T'''(r)}{T''(r)}\sin\frac{r}{2}\right);$$

but (5) implies $T'''(r)/T''(r)>0$, so that $|\delta(r)|>|\delta(r-\pi)|$, where the last symbol has the original meaning. It follows that the root-interval immediately to the right of the interval containing $-\pi$ has a root for T' whose distance from $-\pi$ is greater than r. But an interchange of the rôle of the two regular intervals excludes the possibility $r-\pi<z<0$.

There remains the case where T has two regular intervals (x_1, x_2) and (x_3, x_4) with no endpoint in common. Here the root set of t is specified as in (B1): If none of the x_j coincides with a or b, the latter and z shall

be double roots and the x_j nonroots. If there is coincidence the corresponding root is kept simple.

CONCLUSION. A tp $T \in \tau$ maximizing f has only simple roots, and $|T(x)|$ assumes its maximal value in all roots for T', i.e. $T(x)$ is proportional to a sine function, as postulated.

REFERENCE

1. P. Erdös, *Note on some elementary properties of polynomials*, Bull. Amer. Math. Soc. **46** (1940), 954–958 (see p. 957). MR **2**, 242.

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