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A UNIVERSAL SPACE FOR G-ACTIONS IN WHICH A NORMAL SUBGROUP ACTS FREELY

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ABSTRACT. A universal space is constructed and it is used to show that $\mathfrak{N}_*^G(X, A, \psi)$ is computable when G is a finite supersolvable group.

1. **Background.** This paper follows a suggestion of R. E. Stong [**3**, p. 11]. Let G be a compact Lie group, K a topological group, and $\alpha: G \to \mathrm{Aut}\, K$ a homomorphism from G to the group of automorphisms of K. The image of g by α is denoted by α_g. It is required that $(g, k) \to \alpha_g(k)$ be a continuous map from $G \times K$ into K. Then the following definitions and theorem can be extracted from T. tom Dieck [**4**].

DEFINITION (1.1). A (G, α, K)-*space* is a space W with a continuous left operation, μ, of G on W and a continuous right operation, θ, of K on W such that for every $g \in G$, $k \in K$ and $w \in W$ the following holds:

$$\mu(g, \theta(w, k)) = \theta(\mu(g, w), \alpha_g(k)). \tag{1.1A}$$

DEFINITION (1.2). A (G, α, K)-*bundle* consists of a principal K-bundle $p: E \to B$ where E is a (G, α, K)-space and a continuous left operation of G on B such that p is G-equivariant.

THEOREM (1.3) (T. TOM DIECK). *Let $p: E \to B$ be the universal (G, α, K)-bundle. Then if $\pi: V \to W$ is any numberable (G, α, K)-bundle, there is a bundle map $f: V \to E$ which is (G, K)-equivariant. Any two such bundle maps are homotopic through (G, K)-maps.*

2. **The universal space.** Hereafter suppose that G is a finite group. Let (V, μ) be a G-manifold with boundary. Let K be a normal subgroup of G such that $\mu|_{K \times V}: K \times V \to V$ is free. Define $\alpha: G \to \mathrm{Aut}(K)$ by $g \mapsto (k \to gkg^{-1})$. Define a right action θ: $V \times K \to V$ by $\theta(v, k) = \mu(k^{-1}, v)$. This is a K-action and thus K acts principally. Since K is normal, one can verify condition (1.1A). Thus (V, μ) is a (G, α, K)-space.

PROPOSITION (2.1). *The orbit map $\pi: V \to V/K$ determines a numberable (G, α, K)-bundle.*

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PROOF. Since K acts freely on V one has a principal K-bundle. With the induced action of G on V/K, π is G-equivariant. Therefore, it is a (G, α, K)-bundle. Since V is completely regular and K is a compact Lie group it follows from the work of Palais [2] that the bundle is locally trivial. Furthermore, since V/K is a manifold with boundary the bundle is numberable. □

By Theorem (1.3) there is a (G, K)-equivariant classifying map $f: V \to E$. However, K does not necessarily act freely from the left on E. T. tom Dieck points out that G, α, and K give a $G \times_\alpha K$ semidirect product defined by $(g, k)(g', k') = (gg', \alpha_{g'}(k)k')$. Also $G \times_\alpha K$ operates continuously from the left on V by $\psi((g, k), v) = \theta(\mu(g, v), k)$. Since f is (G, K)-equivariant the following diagram commutes:

$$\begin{array}{ccc} (G \times_\alpha K) \times V & \xrightarrow{\psi} & V \\ {\scriptstyle \mathrm{id}\times f}\downarrow & & \downarrow{\scriptstyle f} \\ (G \times_\alpha K) \times E & \xrightarrow{\psi'} & E \end{array}$$

Thus the classifying map $f: V \to E$ is (G, K)-equivariant on $(G \times_\alpha K)$-actions.

Let $F_K(E)$ denote the fixed set of E by the diagonal action of K where K is thought of as contained in $G \times_\alpha K$ as the pairs (k, k). The above remarks show that $f(V) \subset F_K(E)$.

PROPOSITION (2.2). *Suppose E is a (G, α, K)-space, then $F_K(E)$ is a (G, α, K)-space.*

PROOF. Note that K is normal in $G \times_\alpha K$. Let $y \in F_K(E)$. Suppose that $g \in G$ so $(g, 1) \in G \times_\alpha K$. Then with a slight corruption of notation one has that

$$(k, k)(g, 1)y = (g, 1)(g, 1)^{-1}(k, k)(g, 1)y = (g, 1)(\bar{k}, \bar{k})y = (g, 1)y.$$

So $F_K(E)$ is closed under left action by G. Similarly it is closed under right action by K. □

Suppose that $\mu'(k, e) = e$ for some $e \in F_K(E)$. Then $e = \psi'((k, k), e) = \theta'(\mu'(k, e), k) = \theta'(e, k)$. But E is a principal K-bundle and so K acts freely from the right on E. Thus K acts freely from the left, as a subgroup of G, on $F_K(E)$.

In summary one has the following theorem.

THEOREM (2.3). *There exists a universal space, $F_K(E)$, for G-actions in which a normal subgroup K of G acts freely, and a G-equivariant classifying map $f: (V, \mu, K \text{ free}) \to (F_K(E), \tilde{\mu}, K \text{ free})$. Any two such maps are equivariantly homotopic through G-maps.*

3. **An application.** A *family* $\mathfrak{F}$ in G is a collection of subgroups of G such that: (i) if $H \in \mathfrak{F}$ and $K \subset H$ then $K \in \mathfrak{F}$, and (ii) if $H \in \mathfrak{F}$ and $g \in G$ then $gHg^{-1} \in \mathfrak{F}$. The collection of all subgroups of G, denoted $\mathscr{A}\ell\ell$, is a family. A G-manifold with boundary, (M, μ), is an *$\mathfrak{F}$-free action* if for every $x \in M$, the isotropy group of G at x, $G_x = \{g \in G | \mu(g, x) = x\}$, is in $\mathfrak{F}$.

DEFINITION (3.1). Let (X, A, ψ) be a pair of topological spaces with G-action (given by $\psi: G \times X \to X$ with $\psi(G \times A) \subset A$). A *$G$-bordism element of* (X, A, ψ) is an equivalence class of triples, (M, μ, f), where (M, μ) is a compact G-manifold with boundary and $f: (M, \partial M) \to (X, A)$ is a G-equivariant map. Two triples, (M, μ, f) and (M', μ', f'), are equivalent if there exists a quadruple (V, V^+, ω, F) such that: (i) (V, ω) is a compact G-manifold with boundary, V^+ is a G-invariant submanifold, and $F: (V, V^+) \to (X, A)$ is a G-equivariant map, (ii) $\partial V = M \cup M' \cup V^+$ with $\partial V^+ = (M \cup M') \cap V^+$, $M \cap M' = \varnothing$, $M \cap V^+ = \partial M$ and $M' \cap V^+ = \partial M'$, and (iii) F restricts to f on M and to f' on M', and ω restricts to μ on M and to μ' on M'. Under the operation induced by disjoint union the classes determined by (M, μ, f) for which the dimension of M is n, form a group denoted $\mathfrak{N}_n^G(X, A, \psi)$.

DEFINITION (3.2). Let $\mathfrak{F}$ be any family in G. Then if in Definition (3.1) one requires the compact G-manifolds with boundary to also be $\mathfrak{F}$-free actions one has an *$\mathfrak{F}$-free bordism element of* (X, A, ψ). As before, the equivalence classes determined by compact G-manifolds with boundary of dimension n, under the operation induced by disjoint union form a group, denoted $\mathfrak{N}_n^G(\mathfrak{F})(X, A, \psi)$.

Taking the direct sum over n in the above definitions one obtains the abelian groups, $\mathfrak{N}_*^G(X, A, \psi)$ and $\mathfrak{N}_*^G(\mathfrak{F})(X, A, \psi)$. If N is a closed manifold and one lets $N \cdot (M, \mu, f)$ be equal to $(N \times M, 1 \times \mu, f \circ \pi_M)$ then $\mathfrak{N}_*^G(X, A, \psi)$ and $\mathfrak{N}_*^G(\mathfrak{F})(X, A, \psi)$ are modules over the unoriented bordism ring $\mathfrak{N}_*$. An equivariant map $\Gamma: (X, A, \psi) \to (Y, B, \chi)$ induces homomorphisms

$$\Gamma_* = \mathfrak{N}_*^G(\Gamma): \mathfrak{N}_*^G(X, A, \psi) \to \mathfrak{N}_*^G(Y, B, \psi), \quad \text{and}$$

$$\Gamma_* = \mathfrak{N}_*^G(\mathfrak{F})(\Gamma): \mathfrak{N}_*^G(\mathfrak{F})(X, A, \psi) \to \mathfrak{N}_*^G(\mathfrak{F})(Y, B, \chi)$$

by sending (M, μ, f) to $(M, \mu, \Gamma \circ f)$.

Let K be a normal subgroup of G. The collection $\mathfrak{F}_1 = \{L \subseteq G | L \cap K = \{1\}\}$ is a family. A G-manifold, (M, μ), is an $\mathfrak{F}_1$-free action if and only if K acts freely on M.

The following theorem is an extension of a proposition due to Conner and Floyd [**1**, (19.1)].

THEOREM (3.3). *The $\mathfrak{F}_1$-free bordism group $\mathfrak{N}_*^G(\mathfrak{F}_1)(X, A, \psi)$ is naturally isomorphic to the G/K-bordism group*

$$\mathfrak{N}_*^{G/K}((X \times F_K(E))/K, (A \times F_K(E))/K, (\psi \times \tilde{\mu})^*).$$

PROOF. Let (M, τ, f) represent an element of $\mathfrak{N}_*^G(\mathfrak{F}_1)(X, A, \psi)$. So K acts freely on M and there is the classifying map $c: M \to F_K(E)$. Define $\pi_{f\times c}: M \to X \times F_K(E)$ by $m \mapsto (f(m), c(m))$. $\pi_{f\times c}$ is G-equivariant. K acts freely on $X \times F_K(E)$. $A \times F_K(E)$ is closed under left action by G. Now $\pi_{f\times c}$ induces a G/K-equivariant map $\tilde{\pi}_{f\times c}: M/K \to (X \times F_K(E))/K$. Thus to (M, τ, f) corresponds the triple $(M/K, \bar{\tau}, \bar{\pi}_{f\times c})$. This relation is well defined and so determines a homomorphism, ρ.

The inverse to ρ is constructed as follows. Let (N, η, h) represent an element in $\mathfrak{N}_*^{G/K}((X \times F_K(E))/K, (A \times F_K(E))/K, (\psi \times \tilde{\mu})^*)$. Let π: $X \times F_K(E) \to (X \times F_K(E))/K$ be the orbit map, and let π_X be projection onto X. Consider the following diagram where $\tilde{N}$ is the induced space and $\tilde{h}$ is the map induced by h:

$$\begin{tikzcd} \tilde{N} \arrow[r, dashed, "\tilde{h}"] \arrow[d, dashed] & X \times F_K(E) \arrow[r, "\pi_X"] \arrow[d] & X \\ N \arrow[r, "h"] & (X \times F_K(E))/K. & \end{tikzcd}$$

Note that $\tilde{N}$ is a compact G-manifold with boundary where G acts on $\tilde{N}$ by $\theta(g, (n, (x, e))) = \eta(gK, n), (\psi(g, x), \tilde{\mu}(g, e))$. Furthermore K acts freely on $\tilde{N}$, $\tilde{h}$ is G-equivariant, and $\tilde{h}(\partial\tilde{N}) \subset A \times F_K(E)$. Thus to (N, η, h) associate the triple $(\tilde{N}, \theta, \pi_X \circ h)$. It is immediate that this correspondence is a homomorphism inverse to ρ.

Given an equivariant map $\Gamma: (X, A, \psi) \to (Y, B, \chi)$ it follows immediately from the definitions of Γ_* that the above isomorphism is natural. □

4. **Computing unrestricted bordism groups.** R. E. Stong [**4**, §9] defines the equivariant bordism groups $\mathfrak{N}_*^G(\mathfrak{F})(X, A, \psi)$ to be *computable* if they are naturally isomorphic to a direct sum of ordinary unoriented bordism groups $\mathfrak{N}_{*-k}(Y, B)$, with dimension shifts, as functors on the category of G-pairs. In [**4**, (9.2)] he shows that if G is nilpotent, then $\mathfrak{N}_*^G(\mathscr{A}\ell\ell)(X, A, \psi)$ is computable. Theorem (3.3) allows one to immediately extend two of Stong's propositions [**4**, (8.5) and (3.6)]. One then can obtain the following result.

THEOREM (4.3). *If G is a finite supersolvable group, $\mathfrak{N}_*^G(\mathscr{A}\ell\ell)(X, A, \psi)$ is computable.*

PROOF. If G is a finite supersolvable group then it has a normal series $G = B_0 \supset B_1 \supset \cdots \supset B_n = 1$ such that each factor group B_{i-1}/B_i is cyclic of

prime order, so B_{n-1} is normal of prime order. Now applying the extended Stong propositions one has that the unrestricted G-bordism group of (X, A, ψ) is isomorphic to a direct sum of unrestricted (G/B_{n-1})-bordism groups of some G-pairs. Since subgroups and factor groups of supersolvable groups are supersolvable, G/B_{n-1} is supersolvable. □

Note. This proves computability in a "nice" sequential way. The result in [**4**, §9] for 2 nilpotent groups is stronger yet.

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