PRODUCTS OF ARCWISE CONNECTED SPACES

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Abstract. It is proved that the arbitrary product of arcwise connected spaces is arcwise connected.

Introduction. By an arc we mean a Hausdorff continuum with at most 2 noncut points, called the end points of the arc. A space $S$ is said to be arcwise connected if whenever $x, y \in S$, then $x$ and $y$ are the end points of some arc in $S$. It is well known (see, for instance, [4, Theorems 28.8 and 28.13]) that a nondegenerate metric continuum $A$ is an arc if and only if $A$ is homeomorphic to $[0, 1]$. Since a metrizable product of arcs is a compact, connected and locally connected metric space, it follows [4, Theorem 31.2] that a metrizable product of arcs is arcwise connected. However, examples have been constructed by S. Mardesic [2] and [3] and J. L. Cornette and B. Lehman [1] of locally connected Hausdorff continua which are not arcwise connected. Thus the above argument will not suffice for a nonmetrizable product of arcs, even if each factor space is metrizable. In this paper we show that the arbitrary product of arcwise connected spaces is arcwise connected.

Lemma. Let $\{X_\alpha: \alpha \in \mathcal{A}\}$ be a collection of nondegenerate arcs, and let $X$ denote the product space of this collection. If the end points of $X_\alpha$ are $a_\alpha$ and $b_\alpha$, then there is an arc in $X$ from $f$ to $g$ where $f$ is that point for which $f(\alpha) = a_\alpha$ and $g$ is that point for which $g(\alpha) = b_\alpha$.

Proof. Let $\leq$ be a well-ordering of $\mathcal{A}$, and let "1" denote the first element of $\mathcal{A}$, and "$\alpha + 1$" the successor of $\alpha$ in $\mathcal{A}$. For each $\alpha \in \mathcal{A}$, define the "edge" $A_\alpha$ of $X$ and points $f_\alpha$ and $g_\alpha$ of $X$ as follows:

$$A_\alpha = \{h \in X: h(\beta) = b_\beta, \beta < \alpha; h(\beta) = a_\beta, \alpha < \beta\},$$

$$f_\alpha(\beta) = b_\beta, \quad \beta < \alpha, \quad g_\alpha(\beta) = b_\beta, \quad \beta \leq \alpha,$$

$$= a_\beta, \quad \beta \geq \alpha; \quad = a_\beta, \quad \beta > \alpha.$$
We show that for each $\alpha \in \mathcal{A}$, the following statements are satisfied:

(a) $f_i = f$, and $g_\alpha = f_{\alpha+1}$;
(b) $A_\alpha \cap A_{\alpha+1} = \{g_\alpha\}$;
(c) $A_\alpha$ is an arc homeomorphic to $X_\alpha$ in $X$ and the end points of $A_\alpha$ are $f_\alpha$ and $g_\alpha$;
(d) if $\alpha, \gamma \in \mathcal{A}$ and $\alpha + 1 < \gamma$, then $A_\alpha \cap A_\gamma = \emptyset$;
(e) $\bigcup_{\beta < \alpha} A_\beta \cup \{f_\beta\}$ is an arc in $X$ with end points $f$ and $f_\alpha$. Further, $\bigcup_{\alpha \in \mathcal{A}} A_\alpha \cup \{g\}$ is an arc in $X$ with end points $f$ and $g$.

It is immediate from the definitions that (a) and (b) are satisfied.

If we define a function $\theta : X \to X$ by

$$\theta(x) = \begin{cases} b_\delta, & \delta < \alpha, \\ x, & \delta = \alpha, \\ a, & \delta > \alpha, \end{cases}$$

then $\theta$ is a homeomorphism from $X_\alpha$ onto $A_\alpha$ such that $\theta(a_\alpha) = f_\alpha$, and $\theta(b_\alpha) = g_\alpha$, so (c) is satisfied. Now if $\alpha, \gamma \in \mathcal{A}$, with $\alpha + 1 < \gamma$, and $h \in A_\alpha$, then $h(\alpha + 1) = a_{\alpha + 1} \neq b_{\alpha + 1}$. If $h \in A_\gamma$, then $h(\alpha + 1) = b_{\alpha + 1}$, and it follows that $A_\alpha \cap A_\gamma = \emptyset$. Thus (d) is satisfied.

We now proceed to prove (e) by induction on the well-ordered set $\mathcal{A}$. Let $I(\beta)$ be the statement: $\bigcup_{\alpha < \beta} A_\alpha \cup \{f_\beta\}$ is an arc in $X$ with end points $f$ and $f_\beta$.

If $\beta = 1$, then $\bigcup_{\alpha < 1} A_\alpha \cup \{f_1\} = \{f\}$ so it is an (degenerate) arc with end points $f$ and $f_1$. Suppose that for some $\beta \in \mathcal{A}$, $1 < \beta$, that we have shown that $I(\alpha)$ holds for all $\alpha < \beta$. We consider two cases.

Case 1. $\beta = \gamma + 1$ for some $\gamma \in \mathcal{A}$. By the induction hypothesis $\bigcup_{\alpha < \gamma} A_\alpha \cup \{f_\gamma\}$ is an arc in $X$ with end points $f$ and $f_\gamma$, and we have shown that $A_\gamma$ is an arc in $X$ with end points $f_\gamma$ and $g_\gamma = f_{\gamma + 1} = f_\beta$. If $h \in \bigcup_{\alpha < \gamma} A_\alpha \cap A_\gamma$, then it follows from (d) that $\gamma = \delta + 1$ for some $\delta \in \mathcal{A}$ and $h \in A_\delta \cap A_\gamma$. It then follows from (a) and (b) that $h = g_\delta = f_{\beta + 1} = f_\gamma$, and that $\bigcup_{\alpha < \gamma} A_\alpha \cup \{f_\gamma\} \cap A_\gamma = \{f_\gamma\}$. We have then that $\bigcup_{\alpha < \gamma} A_\alpha \cup \{f_\gamma\}$ and $A_\gamma$ are arcs which meet in a single point, $f_\gamma$, and that $f_\gamma$ is an end point of each. Thus their union is an arc with end points $f_1$ and $g_\gamma = f_\beta$. That is $\bigcup_{\alpha < \gamma} A_\alpha \cup \{f_\beta\}$ is an arc with end points $f$ and $g_\gamma = f_\beta$.

Case 2. $\beta$ has no immediate predecessor in $\mathcal{A}$. There are four steps in the argument.

1. $\bigcup_{\alpha < \beta} A_\alpha$ is connected;
2. If $h \in X$ and $h \notin \bigcup_{\alpha < \beta} A_\alpha \cup \{f_\beta\}$, then $h$ is not a limit point of $\bigcup_{\alpha < \beta} A_\alpha$;
3. $f_\beta$ is a limit point of $\bigcup_{\alpha < \beta} A_\alpha$, so $\bigcup_{\alpha < \beta} A_\alpha \cup \{f_\beta\}$ is a continuum and $f_\beta$ is not a cut point of $\bigcup_{\alpha < \beta} A_\alpha \cup \{f_\beta\}$;
4. $f$ is not a cut point of $\bigcup_{\alpha < \beta} A_\alpha \cup \{f_\beta\}$, and if $h \in \bigcup_{\alpha < \beta} A_\alpha$ and $h \neq f_\beta$,
then $h$ is a cut point of $\bigcup_{\alpha < \beta} A_{\alpha} \cup \{f_\beta\}$; thus $\bigcup_{\alpha < \beta} A_{\alpha} \cup \{f_\beta\}$, is an arc with end points $f$ and $f_\beta$.

**Proof of (1).** By the induction hypothesis, $\bigcup_{\alpha < \beta} A_{\alpha} = \bigcup_{\gamma < \beta} \left( \bigcup_{\alpha < \gamma} A_{\alpha} \right)$ is the union of connected sets each of which contains $f$ and is therefore connected.

**Proof of (2).** For each $\alpha \in \mathcal{A}$ let $P_\alpha$ be the projection map of $X$ onto $X_{\alpha}$. Suppose that $h \in X - \left( \bigcup_{\alpha < \beta} A_{\alpha} \cup \{f_\beta\} \right)$. Let $\gamma$ be the first member of $\mathcal{A}$ such that $h(\gamma) \neq f_\beta(\gamma)$. If $\beta \leq \gamma$ then $f_\beta(\gamma) = a_\gamma \neq h(\gamma)$ and $P_\gamma^{-1}(X_\gamma - \{a_\gamma\})$ is open in $X_\gamma$ contains $h$ and misses $\bigcup_{\alpha < \beta} A_{\alpha} \cup \{f_\beta\}$. Suppose then that $\gamma < \beta$. Then if $\delta < \gamma$, $h(\delta) = b_\delta$ and $h(\gamma) \neq b_\gamma$. If for all $\epsilon \in \mathcal{A}$ such that $\gamma < \epsilon$, $h(\epsilon) = a_\epsilon$, then $h \in A_\gamma$, contrary to assumption. Thus for some $\epsilon \in \mathcal{A}$, $\gamma < \epsilon$ and $h(\epsilon) \neq a_\epsilon$. It now follows from the definition of the sets $A_\alpha$ that the open set $P_\gamma^{-1}(X_\gamma - \{b_\gamma\}) \cap P_\epsilon^{-1}(X_\epsilon - \{a_\epsilon\})$ contains $h$ and misses $\bigcup_{\alpha < \beta} A_\alpha$. Thus $h$ is not a limit point of $\bigcup_{\alpha < \beta} A_{\alpha}$.

**Proof of (3).** We consider the net $\{g_\alpha\}_{\alpha < \beta}$. For all $\delta \in \mathcal{A}$, the net $\{P_\delta(g_\alpha)\}_{\alpha < \beta}$ converges to $P_\delta(f_\beta)$. For if $\beta \leq \delta$, then for all $\alpha < \beta$, $g_\alpha(\delta) = a_\alpha = f_\beta(\delta)$; and if $\delta < \beta$, then since $\beta$ has no immediate predecessor, there is a $\gamma \in \mathcal{A}$, such that $\delta < \gamma < \beta$ and if $\gamma < \epsilon$, then $P_\delta(g_\epsilon) = g_\epsilon(\delta) = b_\delta = f_\beta(\delta)$. It follows that the net $\{g_\alpha\}_{\alpha < \beta}$ converges in $X$ to $f_\beta$. It now follows immediately from (1) that $f_\beta$ is not a cut point of $\bigcup_{\alpha < \beta} A_{\alpha} \cup \{f_\beta\}$.

**Proof of (4).** Since $\bigcup_{\alpha < \beta} A_{\alpha} - \{f\} = \bigcup_{\gamma < \beta} \left[ \bigcup_{\alpha < \gamma} A_{\alpha} - \{f\} \right]$ is a union of the connected sets $\bigcup_{\alpha < \gamma} A_{\alpha} - \{f\}$ each of which contains $f_2$, $\bigcup_{\alpha < \beta} A_{\alpha} - \{f\}$ is connected.

Now suppose that $h \in \bigcup_{\alpha < \beta} A_{\alpha}$, $h \neq f$. Let $\alpha^*$ be the first member of $\mathcal{A}$ such that $h \in A_{\alpha^*}$, and let $Y_{\alpha^*}$, $Z_{\alpha^*}$ be the subarcs in $X_{\alpha^*}$ (one possibly degenerate) with end points $a_{\alpha^*}$, $h(\alpha^*)$ and $h(\alpha^*)$, $b_{\alpha^*}$, respectively. For each $\alpha \in \mathcal{A}$, define $S_\alpha$ and $T_\alpha$ as follows:

$$S_\alpha = \begin{cases} \{b_\alpha\}, & \alpha < \alpha^* \\ Z_{\alpha^*}, & \alpha = \alpha^* \\ X_{\alpha^*}, & \alpha > \alpha^* \end{cases}$$

$$T_\alpha = \begin{cases} X_{\alpha^*}, & \alpha < \alpha^* \\ \{a_\alpha\}, & \alpha > \alpha^* \end{cases}$$

Let $S$ and $T$ be the product spaces respectively of the collections $\{S_\alpha : \alpha \in \mathcal{A}\}$ and $\{T_\alpha : \alpha \in \mathcal{A}\}$. Then $S$ and $T$ are closed in $X$ and $S \cap T = \{h\}$.

Now $f \in T$, $f_\beta \in S$, and $\bigcup_{\alpha < \beta} A_{\alpha} \cup \{f_\beta\} \subset S \cup T$. It follows that $((\bigcup_{\alpha < \beta} A_{\alpha} \cup \{f_\beta\}) - \{h\}) \cap S$, $((\bigcup_{\alpha < \beta} A_{\alpha} \cup \{f_\beta\}) - \{h\}) \cap T$ is a separation of $(\bigcup_{\alpha < \beta} A_{\alpha} \cup \{f_\beta\}) - \{h\}$, so $h$ is a cut point of $\bigcup_{\alpha < \beta} A_{\alpha} \cup \{f_\beta\}$. Thus $I(\beta)$ is established and statement (e) follows.

By argument similar to that in the induction step of the proof of (e), it follows that $\bigcup_{\alpha \in \mathcal{A}} A_{\alpha}$ is connected; that if $h \in X - \bigcup_{\alpha \in \mathcal{A}} A_{\alpha}$ and $h \neq g$, then $h$ is not a limit point of $\bigcup_{\alpha \in \mathcal{A}} A_{\alpha}$; that the net $\{g_\alpha\}_{\alpha \in \mathcal{A}}$ converges in $X$ to $g$, and that $f$ and $g$ are the only noncut points of $\bigcup_{\alpha \in \mathcal{A}} A_{\alpha} \cup \{g\}$. Thus $\bigcup_{\alpha \in \mathcal{A}} A_{\alpha} \cup \{g\}$ is an arc in $X$ with end points $f$ and $g$. 

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Theorem. If \( \{ Y_x : x \in \mathcal{A} \} \) is a collection of arcwise connected spaces, and \( Y \) is the product space of the collection, then \( Y \) is arcwise connected.

Proof. Let \( f, g \) be points of \( Y \). If \( f = g \), there is nothing to prove, so assume that \( f \neq g \). Let \( \mathcal{A}^* = \{ x \in \mathcal{A} : f(x) \neq g(x) \} \). For each \( x \in \mathcal{A}^* \), let \( X_x \) be an arc in \( Y_x \) with end points \( a_x = f(x) \) and \( b_x = g(x) \). Let \( f^* \) and \( g^* \) be the restrictions to \( \mathcal{A}^* \) of \( f \) and \( g \) respectively, and let \( X^* \) be the product space of the collection \( \{ X_x : x \in \mathcal{A}^* \} \). Then \( \{ X_x : x \in \mathcal{A}^* \} \), \( f^* \) and \( g^* \) satisfy the conditions of the Lemma, so there is in \( X^* \) an arc \( A \) with end points \( f^* \) and \( g^* \). Define a map \( \theta : X^* \to Y \) by

\[
[\theta(h)](x) = f(x), \quad x \notin \mathcal{A}^*,
\]

\[
= h(x), \quad x \in \mathcal{A}^*.
\]

Then \( \theta \) is a homeomorphism of \( X^* \) onto \( \theta(X^*) \) with \( \theta(f^*) = f, \theta(g^*) = g \). Thus \( \theta(A) \) is an arc in \( Y \) with end points \( f \) and \( g \).

References


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