PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 1, May 1974

PRODUCTS OF ARCWISE CONNECTED SPACES[1]

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ABSTRACT. It is proved that the arbitrary product of arcwise connected spaces is arcwise connected.

Introduction. By an *arc* we mean a Hausdorff continuum with at most 2 noncut points, called the *end points* of the arc. A space S is said to be *arcwise connected* if whenever $x, y \in S$, then x and y are the end points of some arc in S. It is well known (see, for instance, [**4**, Theorems 28.8 and 28.13]) that a nondegenerate metric continuum A is an arc if and only if A is homeomorphic to [0, 1]. Since a metrizable product of arcs is a compact, connected and locally connected metric space, it follows [**4**, Theorem 31.2] that a metrizable product of arcs is arcwise connected. However, examples have been constructed by S. Mardesic [**2**] and [**3**] and J. L. Cornette and B. Lehman [**1**] of locally connected Hausdorff continua which are not arcwise connected. Thus the above argument will not suffice for a nonmetrizable product of arcs, even if each factor space is metrizable. In this paper we show that the arbitrary product of arcwise connected spaces is arcwise connected.

LEMMA Let $\{X_\alpha : \alpha \in \mathscr{A}\}$ *be a collection of nondegenerate arcs, and let* X *denote the product space of this collection. If the end points of* X_α *are* a_α *and* b_α, *then there is an arc in* X *from* f *to* g *where* f *is that point for which* $f(\alpha)=a_\alpha$ *and* g *is that point for which* $g(\alpha)=b_\alpha$.

PROOF. Let $\leqq$ be a well-ordering of $\mathscr{A}$, and let "1" denote the first element of $\mathscr{A}$, and "$\alpha+1$" the successor of α in $\mathscr{A}$. For each $\alpha \in \mathscr{A}$, define the "edge" A_α of X and points f_α and g_α of X as follows:

$$A_\alpha = \{h \in X : h(\beta) = b_\beta,\ \beta < \alpha;\ h(\beta) = a_\beta,\ \alpha < \beta\},$$

$$\begin{aligned} f_\alpha(\beta) &= b_\beta, & \beta &< \alpha, & \qquad g_\alpha(\beta) &= b_\beta, & \beta &\leqq \alpha, \\ &= a_\beta, & \beta &\geqq \alpha; & &= a_\beta, & \beta &> \alpha. \end{aligned}$$

Received by the editors May 4, 1973.

AMS (MOS) subject classifications (1970). Primary 54B10, 54F05, 54F20.

Key words and phrases. Arc, arcwise connected, product space.

[1] The results in this paper are contained in the author's doctoral dissertation written under the supervision of J. L. Cornette of Iowa State University.

We show that for each $\alpha \in \mathscr{A}$, the following statements are satisfied:

(a) $f_1=f$, and $g_\alpha=f_{\alpha+1}$;

(b) $A_\alpha \cap A_{\alpha+1}=\{g_\alpha\}$;

(c) A_α is an arc homeomorphic to X_α in X and the end points of A_α are f_α and g_α;

(d) if $\alpha, \gamma \in \mathscr{A}$ and $\alpha+1<\gamma$, then $A_\alpha \cap A_\gamma=\varnothing$;

(e) $\bigcup_{\beta<\alpha} A_\beta \cup \{f_\alpha\}$ is an arc in X with end points f and f_α. Further, $\bigcup_{\alpha\in\mathscr{A}} A_\alpha \cup \{g\}$ is an arc in X with end points f and g.

It is immediate from the definitions that (a) and (b) are satisfied. If we define a function $\theta: X_\alpha \to X$ by

$$\begin{aligned}[\theta(x)](\delta) &= b_\delta, & \delta &< \alpha,\\ &= x, & \delta &= \alpha,\\ &= a, & \delta &> \alpha,\end{aligned}$$

then θ is a homeomorphism from X_α onto A_α such that $\theta(a_\alpha)=f_\alpha$, and $\theta(b_\alpha)=g_\alpha$, so (c) is satisfied. Now if $\alpha, \gamma \in \mathscr{A}$, with $\alpha+1<\gamma$, and $h \in A_\alpha$, then $h(\alpha+1)=a_{\alpha+1}\neq b_{\alpha+1}$. If $h \in A_\gamma$, then $h(\alpha+1)=b_{\alpha+1}$, and it follows that $A_\alpha \cap A_\gamma=\varnothing$. Thus (d) is satisfied.

We now proceed to prove (e) by induction on the well-ordered set $\mathscr{A}$. Let $I(\beta)$ be the statement: $\bigcup_{\alpha<\beta} A_\alpha \cup \{f_\beta\}$ is an arc in X with end points f and f_β.

If $\beta=1$, then $\bigcup_{\alpha<\beta} A_\alpha \cup \{f_1\}=\{f\}$ so it is an (degenerate) arc with end points f and f_1. Suppose that for some $\beta \in \mathscr{A}$, $1<\beta$, that we have shown that $I(\alpha)$ holds for all $\alpha<\beta$. We consider two cases.

Case 1. $\beta=\gamma+1$ for some $\gamma \in \mathscr{A}$. By the induction hypothesis $\bigcup_{\alpha<\gamma} A_\alpha \cup \{f_\gamma\}$ is an arc in X with end points f and f_γ, and we have shown that A_γ is an arc in X with end points f_γ and $g_\gamma=f_{\gamma+1}=f_\beta$. If $h \in \bigcup_{\alpha<\gamma} A_\alpha \cap A_\gamma$, then it follows from (d) that $\gamma=\delta+1$ for some $\delta \in \mathscr{A}$ and $h \in A_\delta \cap A_\gamma$. It then follows from (a) and (b) that $h=g_\delta=f_{\delta+1}=f_\gamma$ and that $(\bigcup_{\alpha<\gamma} A_\alpha \cup \{f_\gamma\}) \cap A_\gamma=\{f_\gamma\}$. We have then that $\bigcup_{\alpha<\gamma} A_\alpha \cup \{f_\gamma\}$ and A_γ are arcs which meet in a single point, f_γ, and that f_γ is an end point of each. Thus their union is an arc with end points f_1 and $g_\gamma=f_\beta$. That is $\bigcup_{\alpha\leqq\gamma} A_\alpha \cup \{f_\beta\}$ is an arc with end points f and $g_\gamma=f_\beta$.

Case 2. β has no immediate predecessor in $\mathscr{A}$. There are four steps in the argument.

(1) $\bigcup_{\alpha<\beta} A_\alpha$ is connected;

(2) If $h \in X$ and $h \notin \bigcup_{\alpha<\beta} A_\alpha \cup \{f_\beta\}$, then h is not a limit point of $\bigcup_{\alpha<\beta} A_\alpha$;

(3) f_β is a limit point of $\bigcup_{\alpha<\beta} A_\alpha$, so $\bigcup_{\alpha<\beta} A_\alpha \cup \{f_\beta\}$ is a continuum and f_β is not a cut point of $\bigcup_{\alpha<\beta} A_\alpha \cup \{f_\beta\}$;

(4) f is not a cut point of $\bigcup_{\alpha<\beta} A_\alpha \cup \{f_\beta\}$, and if $h \in \bigcup_{\alpha<\beta} A_\alpha$ and $h\neq f$,

then h is a cut point of $\bigcup_{\alpha<\beta} A_\alpha \cup \{f_\beta\}$; thus $\bigcup_{\alpha<\beta} A_\alpha \cup \{f_\beta\}$, is an arc with end points f and f_β.

Proof of (1). By the induction hypothesis, $\bigcup_{\alpha<\beta} A_\alpha = \bigcup_{\gamma<\beta} (\bigcup_{\alpha<\gamma} A_\alpha)$ is the union of connected sets each of which contains f and is therefore connected.

Proof of (2). For each $\alpha \in \mathscr{A}$ let P_α be the projection map of X onto X_α. Suppose that $h \in X - [\bigcup_{\alpha<\beta} A_\alpha \cup \{f_\beta\}]$. Let γ be the first member of $\mathscr{A}$ such that $h(\gamma) \neq f_\beta(\gamma)$. If $\beta \leqq \gamma$ then $f_\beta(\gamma) = a_\gamma \neq h(\gamma)$ and $P_\gamma^{-1}(X_\gamma - \{a_\gamma\})$ is open in X, contains h and misses $\bigcup_{\alpha<\beta} A_\alpha \cup \{f_\beta\}$. Suppose then that $\gamma < \beta$. Then if $\delta < \gamma$, $h(\delta) = b_\delta$ and $h(\gamma) \neq b_\gamma$. If for all $\varepsilon \in \mathscr{A}$ such that $\gamma < \varepsilon$, $h(\varepsilon) = a_\varepsilon$, then $h \in A_\gamma$, contrary to assumption. Thus for some $\varepsilon \in \mathscr{A}$, $\gamma < \varepsilon$ and $h(\varepsilon) \neq a_\varepsilon$. It now follows from the definition of the sets A_α that the open set $P_\gamma^{-1}(X_\gamma - \{b_\gamma\}) \cap P_\varepsilon^{-1}(X_\varepsilon - \{a_\varepsilon\})$ contains h and misses $\bigcup_{\alpha<\beta} A_\alpha$. Thus h is not a limit point of $\bigcup_{\alpha<\beta} A_\alpha$.

Proof of (3). We consider the net $\{g_\alpha\}_{\alpha<\beta}$. For all $\delta \in \mathscr{A}$, the net $\{P_\delta(g_\alpha)\}_{\alpha<\beta}$ converges to $P_\delta(f_\beta)$. For if $\beta \leqq \delta$, then for all $\alpha < \beta$, $g_\alpha(\delta) = a_\delta = f_\beta(\delta)$; and if $\delta < \beta$, then since β has no immediate predecessor, there is a $\gamma \in \mathscr{A}$, such that $\delta < \gamma < \beta$ and if $\gamma \leqq \varepsilon < \beta$, then $P_\delta(g_\varepsilon) = g_\varepsilon(\delta) = b_\delta = f_\beta(\delta)$. It follows that the net $\{g_\alpha\}_{\alpha<\beta}$ converges in X to f_β. It now follows immediately from (1) that f_β is not a cut point of $\bigcup_{\alpha<\beta} A_\alpha \cup \{f_\beta\}$.

Proof of (4). Since $\bigcup_{\alpha<\beta} A_\alpha - \{f\} = \bigcup_{\gamma<\beta} [\bigcup_{\alpha<\gamma} A_\alpha - \{f\}]$ is a union of the connected sets $\bigcup_{\alpha<\gamma} A_\alpha - \{f\}$ each of which contains f_2, $\bigcup_{\alpha<\beta} A_\alpha - \{f\}$ is connected.

Now suppose that $h \in \bigcup_{\alpha<\beta} A_\alpha$, $h \neq f$. Let α^* be the first member of $\mathscr{A}$ such that $h \in A_{\alpha^*}$, and let Y_{α^*}, Z_{α^*} be the subarcs in X_{α^*} (one possibly degenerate) with end points a_{α^*}, $h(\alpha^*)$ and $h(\alpha^*)$, b_{α^*}, respectively. For each $\alpha \in \mathscr{A}$, define S_α and T_α as follows:

$$\begin{aligned} S_\alpha &= \{b_\alpha\}, & \alpha &< \alpha^*, & T_\alpha &= X_\alpha, & \alpha &< \alpha^*, \\ &= Z_{\alpha^*}, & \alpha &= \alpha^*, & &= Y_{\alpha^*}, & \alpha &= \alpha^*, \\ &= X_\alpha, & \alpha^* &< \alpha, & &= \{a_\alpha\}, & a^* &< \alpha. \end{aligned}$$

Let S and T be the product spaces respectively of the collections $\{S_\alpha : \alpha \in \mathscr{A}\}$ and $\{T_\alpha : \alpha \in \mathscr{A}\}$. Then S and T are closed in X and $S \cap T = \{h\}$. Now $f \in T$, $f_\beta \in S$, and $\bigcup_{\alpha<\beta} A_\alpha \cup \{f_\beta\} \subset S \cup T$. It follows that $([(\bigcup_{\alpha<\beta} A_\alpha \cup \{f_\beta\}) - \{h\}] \cap S, [(\bigcup_{\alpha<\beta} A_\alpha \cup \{f_\beta\}) - \{h\}] \cap T)$ is a separation of $(\bigcup_{\alpha<\beta} A_\alpha \cup \{f_\beta\}) - \{h\}$, so h is a cut point of $\bigcup_{\alpha<\beta} A_\alpha \cup \{f_\beta\}$. Thus $I(\beta)$ is established and statement (e) follows.

By argument similar to that in the induction step of the proof of (e), it follows that $\bigcup_{\alpha\in\mathscr{A}} A_\alpha$ is connected; that if $h \in X - \bigcup_{\alpha\in\mathscr{A}} A_\alpha$ and $h \neq g$, then h is not a limit point of $\bigcup_{\alpha\in\mathscr{A}} A_\alpha$; that the net $\{g_\alpha\}_{\alpha\in\mathscr{A}}$ converges in X to g, and that f and g are the only noncut points of $\bigcup_{\alpha\in\mathscr{A}} A_\alpha \cup \{g\}$. Thus $\bigcup_{\alpha\in\mathscr{A}} A_\alpha \cup \{g\}$ is an arc in X with end points f and g.

THEOREM. *If* $\{Y_\alpha : \alpha \in \mathscr{A}\}$ *is a collection of arcwise connected spaces, and* Y *is the product space of the collection, then* Y *is arcwise connected.*

PROOF. Let f, g be points of Y. If $f=g$, there is nothing to prove, so assume that $f \neq g$. Let $\mathscr{A}^* = \{\alpha \in \mathscr{A} : f(\alpha) \neq g(\alpha)\}$. For each $\alpha \in \mathscr{A}^*$, let X_α be an arc in Y_α with end points $a_\alpha = f(\alpha)$ and $b_\alpha = g(\alpha)$. Let f^* and g^* be the restrictions to $\mathscr{A}^*$ of f and g respectively, and let X^* be the product space of the collection $\{X_\alpha : \alpha \in \mathscr{A}^*\}$. Then $\{X_\alpha : \alpha \in \mathscr{A}^*\}$, f^* and g^* satisfy the conditions of the Lemma, so there is in X^* an arc A with end points f^* and g^*. Define a map $\theta : X^* \to Y$ by

$$\begin{aligned}[\theta(h)](\alpha) &= f(\alpha), \qquad \alpha \notin \mathscr{A}^*, \\ &= h(\alpha), \qquad \alpha \in \mathscr{A}^*.\end{aligned}$$

Then θ is a homeomorphism of X^* onto $\theta(X^*)$ with $\theta(f^*)=f$, $\theta(g^*)=g$. Thus $\theta(A)$ is an arc in Y with end points f and g.

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