AN ENDOMORPHISM RING WHICH IS NOT ORE

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Abstract. An example is given of a finitely generated module over a commutative ring whose endomorphism ring does not have a classical ring of quotients.

Vasconcelos [3] has shown that the endomorphism ring of a finitely generated module over a commutative noetherian ring has a classical ring of quotients. In [1] the author showed that the endomorphism ring of a finitely generated projective module over a commutative ring has a classical ring of quotients. The purpose of this note is to supply an example of a finitely generated module over a commutative ring whose endomorphism ring does not have a classical ring of quotients.

We recall that a ring $A$ has a (right) classical ring of quotients $Q$ if (i) $A \subseteq Q$, (ii) every regular element (nondivisor of zero) of $A$ is invertible in $Q$, and (iii) every element of $Q$ has the form $ab^{-1}$ where $a$ is an element of $A$ and $b$ is a regular element of $A$. A necessary and sufficient condition for a ring $A$ to have a right ring of quotients is for $A$ to satisfy the right Ore condition: Given $a, b$ in $A$, $b$ regular, then there exist $a', b'$ in $A$, $b'$ regular, such that $ab' = ba'$.

Let $R$ be a commutative ring and $I$ an ideal of $R$ such that:
(i) whenever $a \in R$ is regular then $a + i$ is regular for all $i \in I$;
(ii) $\text{Ann}_R(I) = \{x \in R | xI = 0\} = 0$;
(iii) there exists $z \in R$ with $\text{Ann}_R(z) \neq 0$ but $\text{Ann}_R(I, z) = 0$ where $z = z + I \in R/I$. Let $M = R/I \oplus R$. $M$ is clearly a finitely generated $R$-module. We show that $A = \text{End}_R(M)$ does not have a right classical quotient ring. Observe that $A \approx \left[ \begin{array}{c}
\text{End}_R(R/I) & \text{Hom}_R(R, R/I) \\
\text{Hom}_R(R/I, R) & \text{End}_R(R) \end{array} \right]$. Let $f \in \text{Hom}_R(R/I, R)$ then $0 = f(i) = if(1)$ where $i \in I$; $\text{Ann}_R(I) = 0$ implies $f(1) = 0$ and hence $f = 0$. Thus $\text{Hom}_R(R/I, R) = 0$. Also

$\text{End}_R(R/I) \simeq \text{End}_{R/I}(R/I) \simeq R/I$ and $\text{Hom}_R(R, R/I) \simeq R/I$.

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Thus

\[ A \approx \begin{bmatrix} R/I & R/I \\ 0 & R \end{bmatrix}. \]

If \([a \ b]_0\) is regular in \(A\) then \(a\) is regular in \(R/I\) and \(c\) is regular in \(R\). Simple computation shows \([a \ b]_0\) is regular in \(A\), where \(z\) is the element of (iii) above.

We now show that \(A\) does not satisfy the right Ore condition, for if it did then given \([a \ b]_0\) and \([c \ d]_0\) there would exist \([e \ f]_0\) regular in \(A\) and \([g \ h]_0\) in \(A\) such that

\[
\begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix},
\]

which says, in particular, that \(zv = c + i\) for some \(i \in I\). By property (i) \(c + i\) is regular since \(c\) is regular, thus \(zv\) is regular and so \(z\) is regular, a contradiction to the fact that \(\text{Ann}_{R}(z) \neq 0\).

It remains to show that there exists a ring \(R\) with properties (i), (ii), and (iii). For this purpose let \(F\) be a field and \(S = F[X, Y]/(X^2, XY)\) where \(X\) and \(Y\) are (commuting) indeterminants. Let \(N = (X)/(X^2, XY)\), \(y = Y + (X^2, XY) \in S\), and \(\bar{y}\) the image of \(y\) in \(S/N\). We remark that \(S/N \approx F[Y]\) and \(\bar{y}\) corresponds to \(Y\) under the isomorphism. \(\text{Ann}_{S}(y) \neq 0\) since \([Y + (X^2, XY)](X + (X^2, XY)) = (X^2, XY)\). However \(\text{Ann}_{S/N}(\bar{y}) = 0\), since \(\text{Ann}_{R[Y]}(Y) = 0\). Since \(N\) is a nil ideal \(a + n\) is regular for any regular element \(a \in S\) and any \(n \in N\). Since \(N\) is nilpotent we are still missing property (ii). We remedy this as follows: Let \(x_1, x_2, x_3, \ldots\) be a countable set of (commuting) indeterminants and set

\[ R = S[x_1, x_2, \ldots]/(x_1^2, x_2^3, x_3^4, \ldots), \]

where \(S\) is the ring constructed above. Let

\[ I = (N \cdot S[x_1, x_2, \ldots] + (x_1, x_2, \ldots))/(x_1^2, x_2^3, x_3^4, \ldots). \]

Clearly every element of \(I\) is nilpotent and so property (i) holds. \(\text{Ann}_{R}(I) = 0\) since if \(a \in R\) such that \(aI = 0\) then \(a\) must have \(x_1, x_2, x_3, \ldots\) all as factors, an impossibility. Finally let \(z\) be the image in \(R\) of \(y \in S\). Since \(\text{Ann}_{S}(y) \neq 0\), say \(sy = 0\) for \(0 \neq s \in S\), then \(\bar{s} = s + (x_1^2, x_2^3, \ldots) \neq 0\) in \(R\) and \(\bar{z}\bar{s} = 0\), so \(\text{Ann}_{R}(z) \neq 0\). Let \(\bar{z} = z + I \in R/I\); we show \(\text{Ann}_{R/I}(\bar{z}) = 0\). Suppose \(\bar{g} \in R/I\) and \(0 = \bar{g} \bar{z}\) where \(g = s + h + (x_1, x_2, \ldots) \in R\) with \(s \in S\) and \(h \in (x_1, x_2, \ldots)\). Now \(\bar{z}\bar{g} = 0\) implies that \(zg \in I\) that is \(sy + rh \in N \cdot S[x_1, x_2, \ldots] + (x_1, x_2, \ldots)\), which says \(sy \in N\) since \(s, y \in S\). But \(\text{Ann}_{S/N}(\bar{y}) = 0\) so \(s \in N\) and hence \(g \in I\), so \(\bar{g} = 0\) and \(\text{Ann}_{R/I}(\bar{z}) = 0\).
As a final remark we observe that by Nagata’s “principle of idealization” [2] we can form a new ring $R^*$ such that $M$ is an ideal of $R^*$ and also $\text{End}_R(M) \cong \text{End}_{R^*}(M)$. This fact and the last example show that one cannot hope to generalize “finitely generated projective” in [1] to “finitely generated torsionless.”

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