THE EXCESS OF SETS OF COMPLEX EXPONENTIALS

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Abstract. Let \( A = \{\lambda_n\} \) be a complex sequence and denote its associated set of complex exponentials \( \{\exp(i\lambda_n x)\} \) by \( e(A) \). Redheffer and Alexander have shown that if \( \sum |\lambda_n - \mu_m| < \infty \) then \( e(A) \) and \( e(\mu) \) have the same excess over their common completeness interval. This paper shows this result to be the best possible.

1. Introduction. Let \( A = \{\lambda_n\} \) be a complex sequence and denote its associated set of complex exponentials \( \{\exp(i\lambda_n x)\} \) by \( e(A) \). The properties of \( e(A) \) can often be predicted from analyzing the distribution of \( A \), for instance, its completeness interval [2], [6], convergence rates [7], and norm inequalities [4], [5], [8]. In this paper, a condition derived by Redheffer and Alexander [1] which is sufficient for preserving the excess of a set is shown to be the best possible.

Let \( A = \{\lambda_n\} \) be a complex sequence; \( e(A) \) is complete in \( L^2(-a, a) \) if the following condition is satisfied: if \( f \in L^2(-a, a) \) and

\[
\int_{-a}^{a} f(x) \exp(i\lambda_n x) \, dx = 0
\]

for each \( n \), then \( f \equiv 0 \). The interval \( I \) is the completeness interval for \( e(A) \) if the set is complete on all shorter intervals but on no longer intervals. \( e(A) \) has excess \( E(I) \) on an interval if it remains complete when \( E \) terms are removed but not when \( E+1 \) terms are removed. The range of \( E \) may include negative integers as well as \( \pm \infty \) by analogous definitions. The term excess is well defined provided \( A \) satisfies, \( n^m \) implies \( |\lambda_n| \leq |\lambda_m| \) and \( |\lambda_{-n}| \leq |\lambda_{-m}| \) and is regular if \( \inf_{n \neq m} |\lambda_n - \lambda_m| > 0 \).

2. Statement of results. Let \( w(n) \) be a positive weight function defined on the integers and \( \Lambda = \{\lambda_n\} \) be complex sequences.
Theorem I (Redheffer-Alexander [1]). If \( w(n) \geq \delta > 0 \), for all \( n \), then \( \sum |\lambda_n - \mu_n| w(n) < \infty \) implies \( E(\Lambda) = E(U) \).

Theorem II. If \( \inf \{w(n)\} = 0 \), then there exist real, regular, sequences \( \Lambda \) and \( U \) such that \( \sum |\lambda_n - \mu_n| w(n) < \infty \), when \( \Lambda \) is canonically indexed, but \(-\infty < E(\Lambda) < E(U) < \infty \). Theorems I and II characterize the weight functions \( w(n) \) with the property: \( \sum |\lambda_n - \mu_n| w(n) < \infty \) implies \( E(\Lambda) = E(U) \), as those which satisfy \( \inf \{w(n)\} > 0 \).

Theorem II remains valid when \( w(n) \) is allowed to take on the value \(+\infty\), provided \( 0 \cdot \infty = 0 \), and thus no gap theorem can remove the restriction \( \inf \{w(n)\} > 0 \).

3. Proof of Theorem II. We may suppose that \( \inf_{n > 0} \{w(n)\} = 0 \) and let \( n_j \) be the first integer for which \( w(n) \leq j^{-3} \), \( j = 1, 2, \ldots \). Define a sequence of positive integers \( \{k_j\} \) by: \( k_0 = \) arbitrary, large integer and for \( j = 1, 2, \ldots \), \( k_j = \inf \{n \mid n = n_{j+m} \text{ for some } m \geq 0 \text{ and } n \geq k_{j-1}^3 \} \). Theorem II follows from Lemma 3 and Lemma 4. The direct product definition \( F(z) = \prod (1 - z/\lambda_n) \) will be used to designate \( F(z) = \lim_{R \to \infty} \prod |\lambda_n| < R \) \( (1 - z/\lambda_n) \) with convergence easily verified if there is no justification given.

Lemma 3. There is a real, even, regular sequence \( \Lambda \) satisfying \( \{\pm k_j, j \neq 0\} \subset \Lambda \), \( 0 \notin \Lambda \), and \(-\infty < E(\Lambda) < 0 \), such that if \( Q(z) = \prod (1 - z/\lambda_n) \) then \( Q \) is of exponential type \( \pi \), \( Q(x) \in L^2(-\infty, \infty) \), and

\[
\sum j^2 \int_{k_{j+1}}^{k_{j+2}} |Q(x)|^2 \, dx = \infty.
\]

Lemma 4 (Redheffer-Alexander [1]). Let \( \Lambda \) and \( U \) be real sequences with \( 0 \notin \Lambda \), \( U \) and \( n_\lambda(r) \) (\( n_\mu(r) \)) denote the number of terms \( \lambda_n \) (resp. \( \mu_n \)) in the interval \( (0, r) \), counted negatively for negative \( r \), and set \( \Delta r = n_\lambda(r) - n_\mu(r) \). If \( |\Delta r| \leq H \) eventually, then \( |E(\Lambda) - E(U)| \leq 4H+2 \).

Suppose \( \Lambda \) and \( Q(z) \) are as in Lemma 3 and define the sequence \( U \) by:

\[
\mu_n = \lambda_n \quad \text{if } \lambda_n \neq k_j,
\]

\[
k_k - l_j \quad \text{if } \lambda_n = k_j \text{ for some } j,
\]

where \( l_j \) is selected so that \( |l_j - j| \leq 1 \) and the sequence \( U \) remains regular. Lemma 4 holds with \( H = 1 \) so that \( |E(\Lambda) - E(U)| \leq 6 \) and since \( w(n_{j+m}) \leq j^{-3} \), we have \( \sum |\lambda_n - \mu_n| w(n) \leq (j+1)j^{-3} < \infty \).

Set \( P(z) = \prod (1 - z/\mu_n) \) and \( R(z) = P(z)/Q(z) \) so that

\[
R(z) = \prod (1 - l_j z/(k_j - z)(k_j - l_j)).
\]

For a fixed \( z \), let \( k_m \) denote one of the closest \( k_j \) to \( z \) and set

\[
f(z) = |z| \sum_{j \neq m} l_j |k_j - z| (k_j - l_j).
\]
If $|\arg z| \geq \eta > 0$, then $|k_j - z| \geq |z|\sin \eta$, and since $k_j \geq k_{j-1}^2$ for all $j$, for $|\arg z| < \eta$ there is a constant $A$ not depending on $z$ so that $A|k_j - z| \geq |z|$ for $j \neq m$. From the convergence of $\sum l_j|k_j - l_j|^{-1}$, it follows that the series in (1) converges uniformly and that $f(z)$ is uniformly bounded in the complex plane. Thus there is a constant $A'$ so that

$$A' \leq |R(z)| |k_m - z| |k_m - l_m - z| \leq A'^{-1}.$$  

Hence, $P(z)$ has exponential type $\pi$ and

$$\int |P(x)|^2 \, dx \geq A' \sum_{k_j+1}^{k_j+2} \left( Q(x) \frac{(k_j - l_j - x)}{(k_j - x)} \right)^2 \, dx \geq A' \sum_{j=1}^{k_j+2} \frac{(j-1)^2}{2} Q(x)^2 \, dx = \infty.$$  

It must be that $E(U) \geq 0$, for if $-\infty < E(U) < 0$, then by the Paley-Wiener theorem, there would be an entire function $F(z)$ of exponential type $\pi$ which satisfies $F(x) \in L^2(-\infty, \infty)$ and whose zero set is $U \cup \{z_j\}, j=1, 2, \cdots$, $-E(U)-1$, for some nonzero complex numbers $z_j$. By a theorem of Lindelöf, $F(z) = a \exp(bz)P(z)\varphi(1-z/z_j)$ for constants $a$ and $b$. An examination of the indicator functions for $F$ and $P$ shows that $a = 0$ and thus $F(z) \notin L^2(-\infty, \infty)$, a contradiction.

4. Proof of Lemma 3. The sequence $\Lambda$ is constructed from the integers so that $Q(z)$ behaves like $\sin\pi z/\pi z$ except in neighborhoods of the set $\{\pm k_j\}, j=1, 2, \cdots$, where $|Q(x)|$ assumes relatively large values.

Let $\{l_j\}$ be a sequence of positive integers satisfying $l_j \leq k_j^2$ for some $\alpha$, $0 < \alpha < 1$, and $m$ be a fixed positive integer, all to be specified later. For $n > 0$, set

$$\lambda_n = n - m, \quad k_j - l_j \leq n < k_j, \quad k_j - l_j - m \leq n < k_j - l_j,$$

and for $n < 0$, set $\lambda_n = -\lambda_{-n}$. The sequence $\Lambda$ is real, regular, and even with $0 \notin \Lambda$ and $\{\pm k_j\} \subset \Lambda$. For $U = \{n\}, n \neq 0$, Lemma 4 holds with $m = H$ so that $E(\Lambda)$ is finite. For $j = 1, 2, \cdots$, define a sequence of functions $r_{\pm,j}(z)$ by

$$r_{\pm,j}(z) = \prod_{s=k_j-m}^{k_j-1} \left( 1 \pm \frac{z(l_j - \frac{1}{2})}{\lambda_{s-1}(s - z)} \right).$$

For $Q(z) = \pi(1-z/\lambda_n)$, we then have

$$\pi z Q(z) = \sin \pi z \left( \lim_{j \to \infty} \prod_{j < f} r_j(z) \right).$$
The influence of the term \( r_j(z) \) is only local since there is a constant \( A \) so that uniformly in \( z \) and \( j > 0 \), we obtain \( |\ln r_j(z)| \leq A |l^j| k_j \) whenever \( |z-k_j| \geq k_j/2 \). We can assume that \( k_j \geq 2 \) and obtain

\[
|\ln \prod_j |r_j(z)|| \leq 2A \sum 2^{(x-1)/i}, \quad x-1 < 0,
\]

where the ' denotes deletion of those terms for which \( |z-k_j| < k_j/2 \).

From (4) it is clear that \( Q(z) \) has exponential type \( \pi \).

For any real \( x \) satisfying \( |x-k_j| \leq k_j/2 \), the absolute value of each term in (2), \( s=k_j-m, \ldots, k_j-1 \), is dominated by \( \max \{2, 3x|l|/|s-x| \} \).

This bound and the inequality \( |\sin \pi x| \leq \pi |s-x| \) applied to (2), (3) and (4) show that there is a constant \( B \) independent of \( j \) so that

\[
\int_{|z-k_j| \leq k_j/2} |Q(x)|^2 \, dx \leq Bm l^{2m}/|k_j|^2,
\]

and

\[
\int_{2m-1 \leq |z-k_j| \leq k_j/2} |Q(x)|^2 \, dx \leq B \int_{|z-k_j| \leq k_j/2} x^{-2} \, dx
\]

\[
+ \frac{B l^{2m}}{k_j^2} \int_{|z-k_j| \geq m} |x-k_j|^{-2m} \, dx.
\]

Thus, \( Q(x) \in L^2(-\infty, \infty) \) if \( \sum l_j^{2m}/|k_j|^2 < \infty \).

Similarly, \( \sum_{k_j-1}^{k_j-1} |Q(x)|^2 \, dx \geq B \sum l_j^{2m}/|k_j|^2 \) uniformly in \( j \) for some nonzero \( B' \). It is clear that by selecting \( m=2 \) and \( l_j = [(k_j/j)^{1/2}] \), Lemma 3 is satisfied.

5. Remarks and extensions. Without further restrictions on \( w(n) \), the conclusion of Theorem II cannot be altered to give \( \lim \sup \{E(A) - E(U)\} = \infty \).

For instance, setting \( w(n) = j^{-1} \) when \( n=2^j \) and \( w(n)=1 \) otherwise, then \( \sum \hat{\lambda}_n - \mu_n \), \( w(n) < \infty \) implies \( \lim \sup \{E(A) - E(U)\} = \infty \). The convention \( |x-x|=0 \) is used when \( E(A)=\pm \infty \).

To see this, suppose that \( E(A)<0 \) so that there is a nontrivial \( F(z) \) of exponential type \( \pi \), \( F(x) \in L^2(-\infty, \infty) \), whose zero set contains \( \Lambda \).

Without altering the conclusion \( \lim \sup \{E(A) - E(U)\} = \infty \) we may assume \( \hat{\lambda}_n = \mu_n \) when \( n \neq n_j \), \( |\mu_n - \hat{\lambda}_n| \leq j \), and \( |I_m \hat{\mu}_n|, |I_m \hat{\lambda}_n| \geq 1 \), for all \( n \) (see Elsner [3]). Let

\[
Q(z) = F(z) \prod_j \frac{(1 - z/\mu_n)}{(1 - z/\hat{\lambda}_n)}
\]

and suppose that \( \hat{\lambda}_n \) is one of the closest \( \hat{\lambda}_n \) to \( z \). By the estimates in the proof of Theorem II,

\[
A \leq |F(z)| |\mu_n - z|/|Q(z)| |\hat{\lambda}_n - z| \leq A^{-1}
\]
for a constant $A$. Therefore, $Q$ has exponential type $\pi$ and $Q(x)/(1+|x|) \in L^2(-\infty, \infty)$ so that $E(U) < \infty$. By symmetry, $|E(\Lambda) - E(U)| < \infty$.

If we consider only real sequences $\Lambda$ and $U$, Theorem I and Theorem II can be restated in terms of the function $n_\Lambda(r)$. We need only observe that for canonically indexed sequences $\Lambda$ and $U$, if for some $m$, $\int |n_\Lambda(r) - n_U(r) + m| \, dr < \infty$, then $\sum |\lambda_n - \mu_n + m| < \infty$.

**Theorem I'.** Let $w(x)$ be a positive weight function for which $\inf_x \{w(x)\} > 0$. For $\Lambda$ and $U$ are real sequences and for some $m$, $\int |n_\Lambda(r) - n_U(r) + m| w(r) \, dr < \infty$, then $E(\Lambda) = E(U)$.

**Theorem II'.** If $\lim_{x \to \infty} w(x) = 0$, then there are real, regular sequences $\Lambda$ and $U$ such that $\int |n_\Lambda(r) - n_U(r)| w(r) \, dr < \infty$ but $-\infty < E(\Lambda) < E(U) < \infty$.

**References**


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