ON A THEOREM OF A. PEŁCZYŃSKI

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Abstract. If $Y$ is a weakly complete Banach space, and $X$ is a Banach space with separable dual, then every continuous linear operator from $C_\infty(K)$ to $Y$ must be weakly compact. Here $C_\infty(K)$ denotes the space of continuous functions on the compact Hausdorff space $K$, having values in $X$.

In 1953, A. Grothendieck [5] proved that if $Y$ is a weakly complete Banach space, then every continuous linear map from $C(K)$ to $Y$ must be weakly compact. (Here $K$ is an arbitrary compact Hausdorff space.) Later A. Pełczyński [7] weakened the assumption on $Y$, to the requirement that $c_0$ is not isomorphic to any subspace of $Y$. In 1962, Pełczyński [8, p. 645] obtained a result which implies that $C(K)$ may be replaced in the above by $C_X(K)$, the space of continuous $X$-valued functions on $K$, where $X$ is a reflexive Banach space. (See also J. Batt and E. J. Berg [2, p. 237], where a different but related proof is given.) Necessary and sufficient conditions on $X$ for this to still work are not known. However, in order for every continuous linear map $T:C_X(K)\to Y$ to be weakly compact for a given $Y$, it is obviously necessary that every continuous operator from $X$ to $Y$ be weakly compact. But if $X$ has a separable dual space, this holds for weakly complete $Y$, since Cantor's diagonal argument allows us to extract from every bounded sequence in $X$ a subsequence which is weakly Cauchy. This suggests our result below.

Theorem. If $Y$ is a weakly complete Banach space, and $X$ is a Banach space whose separable subspaces have separable duals, then every continuous linear operator from $C_X(K)$ to $Y$ must be weakly compact.

This does not quite include the result of Pełczyński because of our slightly stronger assumption that $Y$ is weakly complete. The interesting thing is that in the above theorem we cannot replace weak completeness of $Y$ by the assumption that $Y$ has no subspace isomorphic to $c_0$. The counterexample is the well-known space $J$ of R. C. James [6], which has
separable bidual, but is not reflexive. (The identity map on \( J \) fails to be weakly compact, even though \( J \) contains no copy of \( c_0 \).)

Batt and Berg's proof of Pelczyński's result proceeds via weak compactness of the adjoint map. This involves a weak compactness theorem in the space of \( X' \)-valued measures, which will not work in our case, as it depends on reflexivity of \( X \). However this approach was used in [1] to prove weak compactness of a map from a \( C^* \)-algebra to a Banach space containing no copy of \( c_0 \). (The details are, needless to say, quite different.)

**Proof of the Theorem.** We first deal with the case that \( K \) and \( X \) (hence also \( X' \)) are separable. By the representation theorem of [2, pp. 225–228], we may represent our map \( T: C_X(K) \to Y \) as an integral with respect to a measure \( \mu \) taking values in the space \([X, Y]\) of bounded operators between \( X, Y \), and having semivariation absolutely continuous with respect to some positive regular measure \( \lambda \) on \( K \). Thus the adjoint \( T' \) maps \( Y' \) into a space of measures with values in \( X' \), all of which are absolutely continuous with respect to \( \lambda \). For such measures the Radon-Nikodym theorem is well known to hold, so we may embed the range of \( T' \) in \( L^1(X', \lambda) \), the space of \( \lambda \)-integrable, \( X' \)-valued functions, by the formula:

\[
\langle f, T'y' \rangle = \langle Tf, y' \rangle = \int f d(y'\mu) = \int f(d(y'\mu)/d\lambda) d\lambda,
\]

for \( f \in C_X(K), y' \in Y' \).

Associated with the element \( y'\mu \) in \( T'Y' \) is the function \( d(y'\mu)/d\lambda \) in \( L^1(X', \lambda) \), and it is easy to see that the embedding of \( T'Y' \) into \( L^1 \) is norm increasing. Since \( L^1(X', \lambda) \) is separable, we conclude that \( T'Y' \) is a separable subspace of the dual of \( C_X(K) \). This fact enables us to use Cantor's diagonal argument to extract a subsequence \( \{f'_{n_m}\} \) such that for every \( y' \) in \( Y' \) the sequence \( \{\langle f'_{n_m}, T'y' \rangle\} = \{\langle Tf'_{n_m}, y' \rangle\} \) is Cauchy. Thus by weak completeness of \( Y \), \( \{Tf'_{n_m}\} \) converges weakly in \( Y \), proving \( T \) is weakly compact.

The reduction to the case \( X, K \), are separable is standard; see for example [2].

**References**


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