DISTORTION PROPERTIES OF $p$-FOLD SYMMETRIC ALPHA-STARLIKE FUNCTIONS

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Abstract. Starlike functions $f$ which are of Mocanu type $\alpha$ and have power series of the form

$$f(z) = z + a_{p+1}z^{p+1} + a_{2p+1}z^{2p+1} + \cdots,$$

where $p=1, 2, 3, \cdots$, are shown to satisfy the relation $f(z) = [g(z^p)]^{1/p}$ where $g$ is of Mocanu type $\rho\alpha$ with power series $g(z) = z + b_2z^2 + b_3z^3 + \cdots$. Distortion results dealing with the $1/4$-theorem and bounds on $|f(z)|$ are obtained.

1. Introduction. In a recent paper [1] S. S. Miller obtained distortion theorems for the class of alpha-starlike functions. In this paper we look at functions which are alpha-starlike and $p$-fold symmetric. Specifically we look at functions $f$ which are alpha-starlike with power series of the form

$$(1.1) \quad f(z) = z + a_{p+1}z^{p+1} + a_{2p+1}z^{2p+1} + \cdots,$$

where $p=1, 2, 3, \cdots$.

For completeness we recall the pertinent definitions.

Definition 1. Let $\alpha$ be real and suppose $f(z) = z + \beta_2z^2 + \beta_3z^3 + \cdots$ is regular in $D = \{z: |z| < 1\}$ with $f(z)f'(z) \neq 0$ in $0 < |z| < 1$. If

$$\text{Re}\left(1 - \alpha \frac{zf''(z)}{f'(z)} + \alpha \left(\frac{zf''(z)}{f'(z)} + 1\right)\right) > 0$$

for $z \in D$ then $f$ is an $\alpha$-starlike function. We write $f \in \mathcal{M}_\alpha$.

Definition 2. If $f$ is starlike and $\alpha = \sup \{\beta: f \in \mathcal{M}_\beta\}$ then $f$ is of Mocanu type $\alpha$ ($f \in \mathcal{M}(\alpha)$).

The above definitions may be found in [1], [2] and [3].

We now introduce some notation.
DEFINITION 3. If \( f \in \mathcal{M}_\alpha \) and \( f(z) \) has a power series of the form (1.1) we write \( f \in \mathcal{M}_{\alpha, \nu} \). If \( f \in \mathcal{M}(\alpha) \) with power series of the form (1.1) we write \( f \in \mathcal{M}_p(\alpha) \).

The results of this paper will depend upon the theorem (proven in §2) that \( f \in \mathcal{M}_p(\alpha) \) iff \( g \in \mathcal{M}_{1}(p\alpha) \) where \( f(z) = [g(z^p)]^{1/p} \). The subsequent distortion theorems (proven in §3) will follow from results in [1].

2. The basic relation. In this section we consider the following result.

THEOREM 1. \( f \in \mathcal{M}_p(\alpha) \) iff \( g \in \mathcal{M}_{1}(p\alpha) \), where \( f(z) = [g(z^p)]^{1/p} \).

PROOF. Let \( f \in \mathcal{M}_{\alpha, \nu}, \) \( \alpha \) real, thus

\[
(2.1) \quad \text{Re}\left\{ \left( 1 - \alpha \right) \frac{zf'(z)}{f(z)} + \alpha \left( \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > 0.
\]

Setting \( f(z) = [g(z^p)]^{1/p} \) and computing \( f'(z)/f(z) \) and \( f''(z)/f'(z) \) we notice the left-hand side of (2.1) is equal to

\[
(2.2) \quad \text{Re}\left\{ \left( 1 - p\alpha \right) \frac{z^pg'(z^p)}{g(z^p)} + p\alpha \left( \frac{z^pg''(z^p)}{g'(z^p)} + 1 \right) \right\}.
\]

But the condition that this quantity is positive is equivalent to \( g \in \mathcal{M}_{p\alpha, 1} \). Since the computations are reversible it follows that \( f \in \mathcal{M}_{\alpha, \nu} \) iff \( g \in \mathcal{M}_{p\alpha, 1} \). Furthermore since the correspondence of \( \alpha \) and \( p\alpha \) is monotone increasing it follows that \( f \in \mathcal{M}_p(\alpha) \) iff \( g \in \mathcal{M}_{1}(p\alpha) \).

Note that an alternate proof to Theorem 1 can be obtained by using the integral representation for functions in \( \mathcal{M}_\alpha \) (see [1] or [2]) plus the fact that if \( g(z) \) is a starlike function then \( [g(z^n)]^{1/n} \) is also a starlike function.

3. Distortion theorems. In the following theorems we will need the functions

\[
(3.1) \quad g_0(p\alpha, z) = \left( \frac{1}{p\alpha} \int_0^z \zeta^{1/p\alpha-1}(1 - \zeta)^{-2/p\alpha} d\zeta \right)^{p\alpha},
\]

\[
(3.2) \quad f_0(\alpha, z) = [g_0(p\alpha, z^p)]^{1/p}
\]

and

\[
(3.3) \quad K(\alpha, r) = r \left[ G\left( \frac{1}{\alpha}, \frac{2}{\alpha}, \frac{1}{\alpha} + 1; r \right) \right]^{\alpha},
\]

where \( G(a, b, c; z) \) is the hypergeometric function.
Theorem 2. If $f(z)$ is a $p$-fold symmetric alpha-starlike function, $\alpha>0$, then for $|z|=r$ ($0<r<1$)

$$[-K(p\alpha, -r^p)]^{1/p} \leq |f(z)| \leq [K(p\alpha, r^p)]^{1/p}. \quad (3.4)$$

Proof. In [1] it is shown that for $g \in \mathcal{M}_1(p\alpha)$,

$$-K(p\alpha, -r) \leq |g(z)| \leq K(p\alpha, r). \quad (3.5)$$

By Theorem 1 $f(z)=[g(z^p)]^{1/p}$ and (3.4) follows. Since (3.5) is sharp for $g_0(p\alpha, z)$, we have equality for $f_0(\alpha, z)$.

Remarks. If $\alpha=1$ and $p=2$ we have for odd convex functions

$$\tan^{-1} r \leq |f(z)| \leq \frac{1}{2} \log \frac{1+r}{1-r},$$

whereas if $\alpha$ approaches zero we have the known result for all odd starlike functions

$$\frac{r}{1+r^2} \leq |f(z)| \leq \frac{r}{1-r^2}.$$}

Furthermore, since $g \in \mathcal{M}_1(\alpha)$ for $\alpha>2$ implies $g$ is a bounded convex function [1] we have that $f \in \mathcal{M}_p(\alpha)$, for $\alpha>2/p$, is a bounded convex function. In particular all odd alpha-starlike functions are bounded if $\alpha>1$.

Theorem 3. If $f \in \mathcal{M}_p(\alpha)$, $\alpha>0$, with power series (1.1) then $|a_{p+1}| \leq 2/p(1+p\alpha)$ and this bound is sharp.

Proof. In [1] it is shown that if $g \in \mathcal{M}_1(p\alpha)$, $p\alpha>0$, the coefficient $b_2=g''(0)/2$ satisfies $|b_2| \leq 2/(1+p\alpha)$. Since $f(z)=[g(z^p)]^{1/p}$, a straightforward calculation shows $|a_{p+1}| \leq 2/p(1+p\alpha)$. This inequality is sharp for $f_0$. Notice that for $\alpha=0$ or $1$ and $p=2$ this reduces to the familiar bounds $1$ and $\frac{1}{2}$ respectively.

Theorem 4. If $f \in \mathcal{M}_p(\alpha)$, $\alpha>0$, then the image of $D$ under the mapping $w=f(z)$ always contains the disc $|w|<\hat{d}(\alpha)$ where

$$\hat{d}(\alpha) = \begin{cases} (\frac{1}{2})^{2/p} & \text{when } \alpha = 0, \\ \left[ \frac{1}{2p\alpha} \frac{[\Gamma(1/p\alpha)]^{2/p}}{\Gamma(2/p\alpha)} \right]^{1/2} & \text{when } \alpha > 0. \end{cases}$$

These results are sharp with equality for $f_0$.

Proof. Clearly $\hat{d}(\alpha)=[d(p\alpha)]^{1/p}$ where $d$ is the radius of the largest disc always contained in the image $w=g(z)$ where $f(z)=[g(z^p)]^{1/p}$. But
by [1],

\[ d(\alpha) = \left[ \frac{1}{\alpha} \frac{\Gamma(1/\alpha)^2}{\Gamma(2/\alpha)} \right]^{\alpha} \]

which proves the result for \( \alpha > 0 \). For \( \alpha = 0 \), the Koebe function gives us \((\frac{1}{2})^{2/\alpha}\) and in fact \( \lim_{\alpha \to 0} \alpha^\alpha = (\frac{1}{2})^{2/\alpha} \), thus establishing the result.

Notice that for \( \alpha = 0 \) or 1 and \( \rho = 1 \) or 2, we have \( \check{d}(\alpha) \) given by

\[
\begin{array}{c|cc}
\alpha & 0 & 1 \\
\hline
p & 1 & 1/2 \\
1 & 1/4 & 1/2 \\
2 & 1/4 & \pi/4 \\
\end{array}
\]

If we let \( p \to \infty \) we notice \( \lim_{p \to \infty} \check{d}(\alpha) = 1 \), \( \alpha \geq 0 \) thus providing another proof of the well-known fact that \( \lim_{p \to \infty} [g(z^p)]^{1/p} \) is the function \( h(z) = z \).

**Theorem 5.** If \( f \in \mathcal{M}_p(\alpha), \ \alpha > 0 \), and \( M(r) = \max_\theta |f(re^{i\theta})| \), then

\[
M(r) = O\left(\frac{1}{1 - r}\right)^{(2 - \alpha)/\rho} \quad \text{for} \quad 0 \leq \alpha < 2/\rho,
\]

\[
= O\left(\log \frac{1}{1 - r}\right) \quad \text{for} \quad \alpha = 2/\rho,
\]

as \( r \to 1^- \). If \( \alpha > 2/\rho \), then

\[
M(r) \leq \left[ \frac{1}{\rho \alpha} \frac{\Gamma(1/\rho \alpha)}{\Gamma(1 - 1/\rho \alpha)} \right]^{\alpha} \cdot \Gamma(1 - 2/\rho \alpha).
\]

These bounds are best possible with equality for \( f_0 \).

**Proof.** From [1] we see that if \( g \in \mathcal{M}_1(\rho \alpha) \), then

\[
\max_\theta |g(r^\rho e^{i\theta})| = O\left(\frac{1}{1 - r}\right)^{2 - \rho \alpha} \quad \text{for} \quad 0 \leq \alpha < 2,
\]

\[
= O\left(\log \frac{1}{1 - r}\right) \quad \text{for} \quad \alpha = 2,
\]

as \( r \to 1^- \). If \( \alpha > 2 \), then

\[
\max_\theta |g(r^\rho e^{i\theta})| \leq \left[ \frac{1}{\rho \alpha} \frac{\Gamma(1/\rho \alpha) \Gamma(1 - 2/\rho \alpha)}{\Gamma(1 - 1/\rho \alpha)} \right]^{\rho \alpha}.
\]

Letting \( f(z) = [g(z^p)]^{1/p} \) and taking \( p \)th roots of the above we obtain the desired result.

**References**


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