ADJOINT FUNCTORS INDUCED BY ADJOINT LINEAR TRANSFORMATIONS

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ABSTRACT. Adjoint linear transformations between Hilbert spaces or, more generally, between dual systems of topological vector spaces induce contravariant functors which are adjoint on the right—essentially a Galois connection between the posets of subsets (or subspaces) of the spaces. Modulo scalars the passage from linear maps to functors is one-to-one; indeed, modulo scalars, two linear transformations are adjoint (hence both are weak continuous) if and only if the induced functors are adjoint.

In a symposium honoring Marshall Stone, Saunders Mac Lane explored a formal analogy of adjoint linear operators on a Hilbert space to adjoint functors between categories [3, pp. 230–232], [4, p. 103]. In contrast, this note singles out a more direct connection: adjoint operators yield adjoint functors.

First, we state the theorem and corollary for Hilbert spaces. Second, as suggested by the referee, we generalize the theorem to dual systems [5, pp. 123, 128] of topological vector spaces and (weak continuous) linear transformations. §3 contains remarks and definitions. §4 proves the general theorem.

1. **Adjoints in Hilbert space.** Consider two complex Hilbert spaces H_0 and H_1, and two maps (=linear transformations) $T_0: H_0 \to H_1$ and $T_1: H_1 \to H_0$. (For notation, let $i, j = 0$ or 1, and $i \neq j$ in the sequel.) The inner product on H_i is denoted by $(\ ,\ )_i$. Let $\mathscr{C}_i$ be the (small) category whose set of objects $|\mathscr{C}_i|$ consists of all subsets of H_i and whose only arrows are inclusions ($X \to Y$ in $\mathscr{C}_i$ iff $X \subseteq Y \subseteq H_i$). For A in $|\mathscr{C}_i|$, let

$$D_i A = \{b \in H_j \mid (T_i a, b)_j = 0, \forall a \in A\}.$$

THEOREM 1. *With T_i and D_i as above the following statements are true.*
(i) *This yields a contravariant functor D_i from $\mathscr{C}_i$ to $\mathscr{C}_j$.*

(ii) *If the maps T_0 and T_1 are adjoint, then the induced contravariant functors D_0 and D_1 are adjoint on the right.*

(iii) *If the induced contravariant functors D_0 and D_1 are adjoint on the right, then there is a scalar λ such that λT_0 and T_1 are adjoint. (If $T_0 \neq 0$, then λ is unique.)*

(iv) *Modulo scalars the passage from maps to functors is one-to-one.*

We state an immediate consequence.

COROLLARY. *If T_0 is selfadjoint, then D_0 is selfadjoint.*

2. **Adjoints in dual systems.** Consider two dual systems E_i, F_i, $(\ , \ )_i$, each consisting of a pair of vector spaces equipped with a nonsingular bilinear form over a (nondiscrete valuated) field K. Consider two linear transformations $T_0 : E_0 \to E_1$ and $T_1 : F_1 \to F_0$. Let $\mathscr{C}_0$ ($\mathscr{C}_1$) be the small category whose set of objects $|\mathscr{C}_0|$ ($|\mathscr{C}_1|$) consists of all subsets of E_0 (F_1), and whose only arrows are inclusions, for example $X \to Y$ in $\mathscr{C}_0$ iff $X \subseteq Y \subseteq E_0$. For $A \in |\mathscr{C}_0|$ and $B \in |\mathscr{C}_1|$, let

$$D_0 A = \{ y \in F_1 \mid (T_0 a, y)_1 = 0, \forall a \in A \},$$

and

$$D_1 B = \{ x \in E_0 \mid (x, T_1 b)_0 = 0, \forall b \in B \}.$$

THEOREM 2. *For dual systems, with T_i and D_i as above, the statements of Theorem 1 hold.*

3. **Definitions, remarks, and examples.** Given a complex Hilbert space H_i, define a dual system by $E_i = H_i$ and F_i equals H_i with the conjugate complex scalar multiplication thereby allowing the bilinear form on $E_i \times F_i$ to be the given Hermitian inner product on H_i. Note that a function $T_1 : H_1 \to H_0$ is linear iff it is linear as $F_1 \to F_0$. Thus the functors D_i of §2 generalize those of §1.

Recall that the pair of maps T_0 and T_1 between Hilbert spaces (dual systems) are *adjoint* [2, p. 39] ([5, p. 128]) if for all (x, y) in $H_0 \times H_1$ (in $E_0 \times F_1$)

(AM) $(T_0 x, y)_1 = (x, T_1 y)_0,$

while two contravariant functors D_0 and D_1 are *adjoint on the right* [1, p. 81] if we have the natural isomorphism of hom-sets

(AF) $\mathscr{C}_1 (Y, D_0 X) \simeq \mathscr{C}_0 (X, D_1 Y),$ $\forall (X, Y) \in |\mathscr{C}_0| \times |\mathscr{C}_1|.$

For Hilbert spaces these are both symmetric relationships (unlike the adjointness of covariant functors). In the special case where $T_0 = T_1 =$ the identity map on H_0, we have $D_0 A = A^\perp$ the *orthogonal complement* of the subset A.

Since the categories $\mathscr{C}_0$ and $\mathscr{C}_1$ are essentially posets, having D_0 and D_1 adjoint on the right is essentially a *Galois connection* [**4**, Theorem 1, p. 93].

All statements and proofs in this note hold if the categories $\mathscr{C}_i$ are replaced by their full subcategories whose objects are closed subspaces and if the singleton subset $\{x\}$ is replaced by its span $\{\lambda x | \lambda \in K\}$.

For the linear transformation $T_0 : E_0 \to E_1$, existence of an adjoint is equivalent to *weak continuity* [**5**, p. 128].

Finally, everything in Theorem 1 is a special case of Theorem 2. So it only remains to prove the latter.

4. Proof of the general theorem. (i) Given the arrow $X \to Y$ in $\mathscr{C}_0$, one verifies $D_0 Y \subseteq D_0 X \subseteq F_1$; similarly, D_1 is functorial.

(ii) If the maps T_0 and T_1 are adjoint, then the equation (AM) holds. We must show (AF) holds. This follows from a reversible chain of implications: $Y \subseteq D_0 X$ implies $\forall (x, y) \in X \times Y$ that $(T_0 x, y)_1 = 0$, which by adjointness (AM) implies $\forall (x, y) \in X \times Y$ that $(x, T_1 y)_0 = 0$, hence $X \subseteq D_1 Y$.

(iii) Given (AF), the left side of (AM) equals 0 iff the right side of (AM) equals 0, since $\{y\} \subseteq D_0\{x\}$ iff $\{x\} \subseteq D_1\{y\}$. Thus $T_0 = 0$ implies (AM) holds, since both sides are 0.

Otherwise pick a vector $x \in E_0$ with $T_0 x \neq 0$, and consider the linear functionals $f = (T_0 x, \quad)_1$ and $g = (x, T_1)_0 : F_1 \to K$. For $y \in F_1$, $fy = 0$ implies $gy = 0$ by (AF). Thus there is a unique scalar $\lambda \in K$ with $g = \lambda f$. It remains to show that

$$(\lambda T_0 a, b) = (a, T_1 b), \forall (a, b) \in E_0 \times F_1.$$

The equation holds if $T_0 a$ is a multiple of $T_0 x$, since $T_0 a = \mu T_0 x$ implies $a - \mu x = n \in \ker T_0$, which implies $(n, T_1 b)_0 = 0$ by (AF). Otherwise, if $T_0 a$ and $T_0 x$ are linearly independent, denote $x' = a$ and $x'' = x' + x$ with corresponding linear functionals and scalars as before satisfying $g' = \lambda' f'$ and $g'' = \lambda'' f''$. Consider the chain of equalities:

$$\lambda' f' + \lambda f = g' + g = g'' = \lambda'' f'' = \lambda'' f' + \lambda'' f.$$

Hence we have $\lambda' = \lambda'' = \lambda$, thus the adjointness equation holds for all $a \in E_0$. *Note.* We have freely used the bilinearity of the bilinear forms and the linearity of the maps, but no continuity or prior existence of adjoints was assumed.

(iv) If the two maps T_0 and T_0' induce the same functor D_0, we will show $T_0' = \lambda T_0$ for some scalar. We may assume T_0' has an adjoint T_1 by enlarging F_0 and F_1 if necessary [**5**, p. 128]. Then by (iii), there is $\lambda \in K$ such that λT_0 is adjoint to T_1. Hence T_0' equals λT_0.

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