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APPROXIMATING FIXED POINTS
OF NONEXPANSIVE MAPPINGS
H. F. SENTER AND W. G. DOTSON, JR.

Abstract. A condition is given for nonexpansive mappings which assures convergence of certain iterates to a fixed point of the mapping in a uniformly convex Banach space. A relationship between the given condition and the requirement of demicompactness is established.

Introduction. Browder [1] and Kirk [7] have shown that a nonexpansive mapping $T$ which maps a closed, bounded, convex subset $C$ of a uniformly convex Banach space into itself has a nonempty fixed point set in $C$. In general, however, for arbitrary $x \in C$ the Picard iterates $T^n x$ do not converge to a fixed point of $T$. It will be shown that if $T$ satisfies one additional condition, then an iterative process of the type introduced by W. R. Mann [8] converges to a fixed point of $T$. For nonexpansive mappings $T$ which have fixed points, this additional condition is weaker than the requirement that $T$ be demicompact.

Convergence to a fixed point. Let $X$ be a Banach space with norm $| \cdot |$ and $C$ a convex subset of $X$. A self-mapping $T$ of $C$ is said to be nonexpansive provided $|Tx - Ty| \leq |x - y|$ holds for all $x, y \in C$. A mapping $T : C \to C$ with nonempty fixed point set $F$ in $C$ will be said to satisfy Condition I if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$, such that $|x - Tx| \geq f(d(x, F))$ for all $x \in C$, where $d(x, F) = \inf \{|x - z| : z \in F\}$.

Let $P$ denote the set of positive integers. For $x \in C$, $M(x, t_n, T)$ is the sequence $\{x_n\}$ defined by $x_{n+1} = (1-t_n)x_n + t_n Tx_n$ where $t_n \in [a, b]$ for all $n \in P$ and $0 < a < b < 1$. This iterative process has been previously investigated by Dotson in [4].

Our main result for nonexpansive mappings is the following:

Theorem 1. Suppose $X$ is a uniformly convex Banach space, $C$ is a closed, bounded, convex, nonempty subset of $X$, and $T$ is a nonexpansive mapping of $C$ into $C$. Let $F$ denote the fixed point set of $T$ in $C$, and suppose

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T satisfies Condition I. Then for any \( x_1 \in C \), \( M(x_1, t_n, T) \) converges to a member of \( F \).

Given that \( F \) is nonempty (which in Theorem 1 is assured by the Browder-Kirk theorem [1], [7]) the proof that \( M(x_1, t_n, T) \) converges to a fixed point uses only the fact that \( T \) is nonexpansive about its fixed points (see Theorem 2 below). Theorem 1 will follow immediately as a corollary of Theorem 2. As in [3], a self-mapping \( T \) of \( C \) will be called quasi-nonexpansive provided \( T \) has a fixed point in \( C \) and if \( p \in C \) is a fixed point of \( T \) then \( |Tx-p| \leq |x-p| \) is true for all \( x \in C \). The class of quasi-nonexpansive mappings includes continuous as well as discontinuous mappings which are not nonexpansive. One can easily prove that \( T: C \to C \) is quasi-nonexpansive if \( T \) has a fixed point in \( C \) and for \( x, y \in C \) satisfies either

\[
(A) \quad |Tx - Ty| \leq \beta (|x - Tx| + |y - Ty|), \quad 0 \leq \beta \leq 1/2,
\]

or

\[
(B) \quad |Tx - Ty| \leq a |x - Tx| + b |y - Ty| + c |x - y|,
\]

where \( a, b, c > 0 \) and \( a + b + c \leq 1 \).

Mappings which satisfy the requirement (A) or (B) have been recently investigated by Kannan [6] and Reich [12] respectively.

For a uniformly convex Banach space \( X \), Dotson [4] has shown that if \( \{w_n\} \) and \( \{y_n\} \) are sequences in the closed unit ball of \( X \) and if \( \{z_n\} = \{(1-t_n)w_n + t_n y_n\} \) satisfies \( \lim |z_n| = 1 \), where \( t_n \in [a, b] \) for \( 0 < a < b < 1 \), then \( \lim |w_n - y_n| = 0 \). This result will be used in the proof of

**Theorem 2.** Suppose \( X \) is a uniformly convex Banach space, \( C \) is a closed, convex subset of \( X \) and \( T \) is a quasi-nonexpansive mapping of \( C \) into \( C \). If \( T \) satisfies Condition I, where \( F \) is the fixed point set of \( T \) in \( C \), then for arbitrary \( x_1 \in C \), \( M(x_1, t_n, T) \) converges to a member of \( F \).

**Proof.** If \( x_1 \in F \) the result is trivial, so we assume \( x_1 \in C \sim F \). For arbitrary \( z \in F \) we have for \( n \in P \) that \( |Tx_n - z| \leq |x_n - z| \) and so

\[
|x_{n+1} - z| \leq (1 - t_n) |x_n - z| + t_n |Tx_n - z| \leq |x_n - z|.
\]

Thus, \( d(x_{n+1}, F) \leq d(x_n, F) \) for all \( n \in P \). The sequence \( \{d(x_n, F)\} \) is nonincreasing and bounded below, so \( \lim d(x_n, F) \) exists. We now show (indirectly) that this limit must be zero, and in turn, that \( \{x_n\} \) converges to a member of \( F \).

Suppose \( \lim d(x_n, F) = b > 0 \). Then for \( z_0 \in F \), \( \lim |x_n - z_0| = b' \geq b > 0 \). Choose \( N > 0 \) such that \( |x_n - z_0| \leq 2b' \) for \( n \geq N \). If we let \( y_n = (Tx_n - z_0)/|x_n - z_0| \) and \( w_n = (x_n - z_0)/|x_n - z_0| \), then \( |y_n| \leq 1 \) and \( |w_n| = 1 \).
for all $n \in P$; and for $n \geq N$

$$|w_n - y_n| = \frac{|x_n - Tx_n|}{|x_n - z_0|} \leq \frac{f(d(x_n, F))}{|x_n - z_0|} \leq \frac{f(b)}{2b'} > 0.$$ 

Therefore, $\lim |w_n - y_n| \neq 0$. Moreover

$$\lim |(1 - t_n)w_n + t_n y_n| = \lim |x_{n+1} - z_0|/|x_n - z_0| = b'/b' = 1.$$ 

However, by the contrapositive of Dotson’s result [4] stated above, since $\lim |w_n - y_n| \neq 0$ then the existence of $\lim |(1 - t_n)w_n + t_n y_n|$ implies $\lim |(1 - t_n)w_n + t_n y_n| \neq 1$, a contradiction. Therefore, $\lim d(x_n, F) = 0$. We show that this implies $\{x_n\}$ converges to an element of $F$.

Since $\lim d(x_n, F) = 0$, given $\varepsilon > 0$ there exists $N_0 > 0$ and $z_0 \in F$ such that $|x_n - z_0| < \varepsilon$, which implies $|x_n - z_0| < \varepsilon$ for all $n \geq N_0$. Thus, if $\varepsilon_k = 1/2^k$ for $k \in P$, then corresponding to each $\varepsilon_k$ there is an $N_k > 0$ and a $z_k \in F$ such that $|x_n - z_k| < \varepsilon_k/4$ for all $n \geq N_k$. We require $N_{k+1} \geq N_k$ for all $k \in P$. We have for all $k \in P$

$$|z_k - z_{k+1}| = |z_k - x_{N_{k+1}} + x_{N_{k+1}} - z_{k+1}| < \varepsilon_k/4 + \varepsilon{k+1}/4 = 3\varepsilon_{k+1}/4.$$ 

Let $S(z, \varepsilon) = \{x \in X: |x - z| \leq \varepsilon\}$ denote the closed sphere centered at $z$ of radius $\varepsilon$. For $x \in S(z_{k+1}, \varepsilon_{k+1})$ we have

$$|z_k - x| = |z_k - z_{k+1} + z_{k+1} - x| < 3\varepsilon_{k+1}/4 + \varepsilon_{k+1} < 2\varepsilon_{k+1} = \varepsilon_k.$$ 

That is, $S(z_{k+1}, \varepsilon_{k+1}) \subseteq S(z_k, \varepsilon_k)$ for $k \in P$. Thus, $\{S(z_k, \varepsilon_k)\}$ is a nested sequence of nonvoid closed spheres with radii $\varepsilon_k$ tending to zero. By the Cantor intersection theorem, $\bigcap_{k \in P} S(z_k, \varepsilon_k)$ contains exactly one point, say $w$. The fixed point set $F$ is closed by [3] and the sequence $\{z_k\}$ from $F$ converges to $w$, so $w \in F$. Since $|x_n - z_k| < \varepsilon_k/4$ for $n \geq N_k$, we have $\{x_n\} \rightarrow w$. Q.E.D.

Note that in Theorem 2 the set $C$ is not required to be bounded; however, boundedness of $C$ is needed in Theorem 1 to apply the Browder-Kirk theorem.

In the preceding theorems, the fixed point of $T$ to which $M(x_1, t_n, T)$ converges depends, in general, on the initial approximation $x_1$ as well as the values of the $t_n$. Also, $M(x_1, t_n, T)$ need not converge to the fixed point of $T$ nearest $x_1$. The following example can be used to verify each of these facts. Let $X$ be the space $R^2$ with the Euclidean norm and, with $(r, \theta)$ denoting polar coordinates, let $C = \{(r, \theta): 0 \leq r \leq 1, -\pi/2 \leq \theta \leq -\pi/4\}$. Define $T: C \rightarrow C$ by $T((r, \theta)) = (r, -\pi/2)$ for each point $(r, \theta)$ in $C$. The set of fixed points of $T$ is the line segment $F = \{(r, -\pi/2): 0 \leq r \leq 1\}$. 

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On Condition I. If \( T: C \to C \) has a nonvoid fixed point set \( F \), then \( T \) will be said to satisfy Condition II provided there exists a real number \( \alpha > 0 \) such that \( |x - Tx| \geq \alpha \cdot d(x, F) \) holds for all \( x \in C \), where as before \( d(x, F) = \inf_{z \in F} |x - z| \). Clearly mappings which satisfy Condition II also satisfy Condition I, and in some cases Condition II is easily verified. In the example above, Condition II holds with \( \alpha = 1 \). If \( T \) rotates points of the unit ball of \( \mathbb{R}^2 \) through an angle \( \pi/2 \), then Condition II holds with \( \alpha = \sqrt{2} \).

Condition II is similar to, but less restrictive than, a requirement imposed by Outlaw in [10, Theorem 2]. Mappings satisfying Outlaw's condition can have at most a single fixed point; his second theorem follows as a special case of Theorem 2.

If \( T: C \to C \) satisfies either requirement (A) or (B) (see above) and has a fixed point in \( C \), then it is easily shown that \( T \) has a unique fixed point \([6], [12]\). In [6, Theorem 2] Kannan proves under certain conditions that for \( x_1 \in C \), \( M(x_1, \frac{1}{2}, T) \) converges to the fixed point of \( T \) if \( T \) satisfies (A). We extend his result with

**Theorem 3.** Let \( C \) be a subset of a Banach space \( X \) and \( T \) a mapping of \( C \) into \( C \) which satisfies either (A) or (B) and has a (unique) fixed point in \( C \). Then \( T \) satisfies Condition II. If \( C \) is closed and convex and \( X \) is uniformly convex then for any \( x_1 \in C \), \( M(x_1, t_n, T) \) converges to the fixed point of \( T \).

**Proof.** Assume \( T \) satisfies requirement (B) and let \( p \) be the unique fixed point of \( T \). Then for \( x \in C \)

\[
|Tx - p| = |Tx -Tp| \leq a |x - Tx| + c |x - p|
\]

and

\[
|Tx - p| \geq |Tx - x| - |x - p| \geq |x - p| - |x - Tx|.
\]

Hence

\[
a |x - Tx| + c |x - p| \geq |x - p| - |x - Tx|,
\]

so \( |x - Tx| \geq [(1-c)/(1+a)]|x - p| \). The constant \( (1-c)/(1+a) \) is positive since \( 0 < a, c < 1 \). Thus Condition II holds. A similar argument applies if \( T \) is a mapping of the type (A).

Since \( T \) is quasi-nonexpansive and satisfies Condition I, the second assertion of the theorem follows directly from Theorem 2. Q.E.D.

We now establish a relationship between mappings which satisfy Condition I and those which are demicompact, beginning with

**Lemma 1.** Suppose \( C \) is a closed, bounded subset of a Banach space \( X \) and \( T: C \to C \) has a nonempty fixed point set \( F \) in \( C \). If \( I - T \) maps closed bounded subsets of \( C \) onto closed subsets of \( X \), then \( T \) satisfies Condition I on \( C \).
PROOF. Let $M = \sup \{d(x, F): x \in C \}$. If $M = 0$ then $F = C$ and Condition I follows trivially. Suppose $M > 0$; for $0 < r < M$ define $C_r = \{x \in C: d(x, F) \geq r \}$ and $f(r) = \inf \{\|x - Tx\|: x \in C_r \}$. Note that each set $C_r$ is non-empty, closed and bounded. We prove that for arbitrary $r$, $0 < r < M$, $f(r) > 0$.

By hypothesis, $(I - T)C_r = \{x - Tx: x \in C_r \}$ is closed. If $\theta \in (I - T)C_r$ then $\theta = z - Tz$ for some $z \in C_r$, which implies $z = Tz$ and thus $z \in F$. But $d(z, F) \geq r > 0$, a contradiction. Therefore, $\theta \notin (I - T)C_r$. Suppose now that $f(r) = 0$. Then there is a sequence $\{x_n\} \subseteq C_r$ such that $\|x_n - Tx_n\| \to 0$ and hence $\{x_n - Tx_n\} \to \theta$. But $\{x_n - Tx_n\} \subseteq (I - T)C_r$, a closed set. Thus we obtain $\theta \in (I - T)C_r$, contradicting our statement above that $\theta \notin (I - T)C_r$. Therefore, $f(r) > 0$ for $0 < r < M$.

We extend the domain of $f$ to $[0, \infty)$ by defining $f(0) = 0$ and $f(r) = \sup \{f(s): s < M \}$ for $r \geq M$. It is easy to verify that $f$ so defined fulfills the hypotheses of Condition I; in particular, $\|x - Tx\| \geq f(d(x, F))$ for each $x \in C$. Q.E.D.

A consequence of Lemma 1 and Theorem 2 is

**Corollary 1 (Browder and Petryshyn [2]).** Let $C$ be a closed, convex subset of a uniformly convex Banach space $X$ and $T: C \to C$ a non-expansive mapping. For $\lambda \in (0, 1)$ let $T_\lambda$ be given by $T_\lambda = \lambda I + (1 - \lambda)T$. (Notice that $M\{x_1, 1 - \lambda, T\} = \{T^n_\lambda x_1\}$.) If $I - T$ maps closed bounded subsets of $C$ onto closed subsets of $X$ and if the set $F$ of fixed points of $T$ is nonempty, then for any $\lambda \in (0, 1)$ and every $x$ in $C$ the sequence $\{T_\lambda^n x\}$ converges to a member of $F$.

A mapping $T: C \to X$ of a subset $C$ of a Banach space $X$ is said to be demicompact [11] provided whenever $\{x_n\} \subseteq C$ is bounded and $\{x_n - Tx_n\}$ converges then there is a subsequence $\{x_{n_k}\}$ which converges. If a mapping $T$ is continuous as well as demicompact then, according to Opial [9, p. 41], the mapping $I - T$ maps closed bounded subsets of $C$ onto closed subsets of $X$. In particular, if $T: C \to C$ is nonexpansive and demicompact and has a fixed point in $C$, it follows from Opial's result and Lemma 1 that $T$ must satisfy Condition I. Using a different approach, Groetsch [5] has established the convergence of mean-value iterates of nonexpansive, demicompact mappings to a fixed point of the mapping.

**References**


Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27607