SOME PROPERTIES OF SELF-INVERSIVE POLYNOMIALS

P. J. O’HARA AND R. S. RODRIGUEZ

Abstract. A complex polynomial is called self-inversive [5, p. 201] if its set of zeros (listing multiple zeros as many times as their multiplicity indicates) is symmetric with respect to the unit circle. We prove that if \( P \) is self-inversive and of degree \( n \) then \( \|P'\| = \frac{1}{n} \|P\| \) where \( \|P'\| \) and \( \|P\| \) denote the maximum modulus of \( P' \) and \( P \), respectively, on the unit circle. This extends a theorem of P. Lax [4]. We also show that if \( P(z) = \sum_{j=0}^{n} a_j z^j \) has all its zeros on \( |z| = 1 \) then \( 2 \sum_{j=0}^{n} |a_j|^2 \leq \|P\|^2 \). Finally, as a consequence of this inequality, we show that when \( P \) has all its zeros on \( |z| = 1 \) then \( 2^{1/2} |a_n| \leq \|P\| \) and \( 2 |a_j| \leq \|P\| \) for \( j \neq n/2 \). This answers in part a question presented in [3, p. 24].

1. Main theorems. We begin with a

Definition. A polynomial \( P \) with zeros \( z_1, z_2, \ldots, z_n \) is self-inversive if \( \{z_1, z_2, \ldots, z_n\} = \{1/z_1, 1/z_2, \ldots, 1/z_n\} \).

Some properties of self-inversive polynomials are given by the following lemmas. These properties have been noted by other authors (see for example [1] and [5, p. 204]). In what follows, if \( P(z) = \sum_{j=0}^{n} a_j z^j \) then \( \overline{P}(z) \) denotes \( \sum_{j=0}^{n} \overline{a_j} z^j \).

Lemma 1. If \( P(z) = \sum_{j=0}^{n} a_j z^j, a_n \neq 0, \) then the following are equivalent:

(i) \( P \) is self-inversive.
(ii) \( a_n P(z) = a_0 z^n \overline{P}(1/z) \) for each complex number \( z \).
(iii) \( a_0 \overline{a_j} = a_n a_{n-j} ; j = 0, 1, \ldots, n \).

This lemma follows easily from the previous definition.

Lemma 2. If \( P \) is self-inversive and \( P(z) = \sum_{j=0}^{n} a_j z^j, a_n \neq 0, \) then

(i) \( \overline{a_n} [nP(z) - zP'(z)] = a_0 z^{n-1} \overline{P'}(1/z) \) for each \( z \),

and

(ii) \( |nP(z)|/zP'(z) - 1| = 1 \) for each \( z \) on \( |z| = 1 \).

Proof. By the previous lemma we can write: \( \overline{a_n} P(z) = a_0 z^n \sum_{j=0}^{n} \overline{a_j} z^{-j} \).

We obtain (i) by differentiating this last identity. Then (ii) follows from
(i) by noting that $1/z = z$ when $|z| = 1$, and $|a_0| = |a_n|$. We remark that Lemma 2 implies that if $P$ is self-inversive then $P'$ has no zeros on $|z| = 1$ except at the multiple zeros of $P$, a result that is proved by other means in [5, p. 205].

By a circular region is meant the image of the unit disk (open or closed) under a bilinear transformation. We shall need the following theorem of DeBruijn [2].

**Theorem.** Let $K$ be a circular region and let $P$ be any polynomial of degree $n$. If $u \in K$ and $Q(z) = n^{-1}[nP(z) + (u - z)P'(z)]$ then $Q(K) \subseteq P(K)$.

Throughout the rest of this paper we shall use the notation $\|P\|$, $P$ a complex polynomial, to denote the maximum modulus of $P$ on the unit circle. The next theorem extends the result of P. Lax given in [4].

**Theorem 1.** If $P$ is a self-inversive polynomial of degree $n$ then $\|P'\| = \frac{1}{2}n\|P\|$. 

**Proof.** Let $e$ be a point on $|z| = 1$ such that $\|P'\| = |P'(e)|$. Choose $u$ on $|z| = 1$ so that $nP(e) - eP'(e)$ and $uP'(e)$ have the same argument. Then by DeBruijn’s theorem we have

$$nP(e) - eP'(e) + uP'(e) \leq n \|P\|,$$

and hence

$$|nP(e) - eP'(e)| + |P'(e)| \leq n \|P\|.$$

By Lemma 2, $|nP(e) - eP'(e)| = |P'(e)|$, and so $2\|P'\| \leq n\|P\|$. To reverse this inequality we again use Lemma 2 to obtain that if $|z| = 1$ then $n|P(z)| = 2|P'(z)|$ and hence $n\|P\| \leq 2\|P'\|$.

Next we consider the following conjecture presented in [3, p. 24]: If $P$ has all its zeros on $|z| = 1$ and $P(z) = \sum_{j=0}^{n} a_j z^j$ then $2|a_j| \leq \|P\|$ for $j = 0, 1, \ldots, n$. We prove the conjecture when the degree $n$ is odd; when $n$ is even we show that $2|a_j| \leq \|P\|$ for $j \neq n/2$, and $2^{1/2}|a_{n/2}| \leq \|P\|$. In the final section we show that the estimate, $2|a_{n/2}| \leq \|P\|$ for $n$ even, is equivalent to the above conjecture. The validity of this estimate is not established by this paper; however, some partial results are presented.

We need

**Theorem 2.** If $P$ is self-inversive and of degree $n$, and $\sum_{j=-\infty}^{\infty} c_j z^j$ is the Laurent expansion about 0 of $nP(z)/zP'(z)$ in some annulus that contains the unit circle (see the remark following Lemma 2), then

$$\frac{1}{2\pi} \int_{0}^{2\pi} \left| \frac{nP(e^{i\theta})}{P'(e^{i\theta})} \right|^2 d\theta = 2 \text{Re}(c_0).$$
In particular if all the zeros of \( P \) lie on \( |z| = 1 \) then \( c_0 = 1 \) and

\[
\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{nP(e^{i\theta})}{P'(e^{i\theta})} \right|^2 d\theta = 2.
\]

**Proof.** Multiplying (i) of Lemma 2 by \( P(z) \) and then using (ii) of Lemma 1, we obtain that

\[
|nP(z)/P'(z)|^2 = 2 \text{ Re}[nP(z)/zP'(z)] \quad \text{for } |z| = 1.
\]

The first part of the theorem now follows since

\[
c_0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{nP(e^{i\theta})}{P'(e^{i\theta})} d\theta.
\]

When \( P \) has all its zeros on the unit circle then the Gauss-Lucas theorem implies that \( P' \) has all its zeros in \( |z| \leq 1 \). Therefore the function defined by \( nP(z)/[zP'(z)] \) is analytic in \( |z| \geq 1 \) and at \( z = \infty \), and hence \( c_0 = \lim_{z \to \infty} nP(z)/[zP'(z)] = 1 \).

**Corollary 1.** If \( P \) is self-inversive and \( P(z) = \sum_{j=0}^{n} a_jz^j, \ a_n \neq 0 \), then

\[
2 \sum_{j=0}^{n} |a_j|^2 \leq \|P\|^2 \text{ Re}(c_0). \quad \text{In particular if } P \text{ has all its zeros on } |z| = 1 \text{ then } 2 \sum_{j=0}^{n} |a_j|^2 \leq \|P\|^2. \quad \text{Moreover these inequalities are equalities if and only if the zeros of } P \text{ are rotations of the } n \text{th roots of unity.}
\]

**Proof.** By applying Parseval's identity and Theorem 1, we obtain:

\[
\sum_{j=0}^{n} |a_j|^2 = \frac{1}{2\pi} \int_0^{2\pi} |P'(e^{i\theta})|^2 \left| \frac{P(e^{i\theta})}{P'(e^{i\theta})} \right|^2 d\theta \leq \left[ \frac{\|P\|^2}{n} \right] \left[ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{nP(e^{i\theta})}{P'(e^{i\theta})} \right|^2 d\theta \right] = \frac{1}{n} \|P\|^2 \text{ Re}(c_0).
\]

Clearly the inequality above is equality if and only if \( |P'(z)| \) is constant for \( |z| = 1 \) or, in other words, if and only if \( P(z) = a_0 + a_nz^n \) where (since \( P \) is self-inversive) \( |a_0| = |a_n| \).

Using Corollary 1 we now prove the following theorem which answers in the previously mentioned conjecture.

**Theorem 3.** If \( P \) has all its zeros on \( |z| = 1 \) and \( P(z) = \sum_{j=0}^{n} a_jz^j, \ a_n \neq 0 \), then

\[
2|a_j| \leq \|P\| \quad \text{for each } j \neq n/2 \text{ and } 2^{1/2} |a_{n/2}| \leq \|P\|.
\]

**Proof.** From (iii) of Lemma 1 we get that \( |a_j| = |a_{n-j}|, \ j = 0, 1, \ldots, n \). Therefore if \( j \neq n/2 \) then

\[
4|a_j|^2 = 2(|a_j|^2 + |a_{n-j}|^2) \leq 2 \sum_{j=0}^{n} |a_j|^2 \leq \|P\|^2.
\]

For \( n \) even, the estimate, \( 2^{1/2} |a_{n/2}| \leq \|P\| \), also follows immediately from Corollary 1.
2. Remarks concerning the middle coefficient. Throughout this section \( P \) will denote an arbitrary self-inversive polynomial (unless further restrictions are noted) with \( P(z) = \sum_{j=0}^{n} a_{j}z^{j}, \ a_{n} \neq 0 \). Also if \( P \) has all its zeros on \(|z|=1\) and \( n \) is even, then we shall refer to the estimate, \( 2|a_{n/2}| \leq \|P\| \), as the middle coefficient conjecture. We present here a few remarks which are pertinent to this conjecture.

A. By using Lemma 1 it is not hard to show that the modulus of the middle coefficient of \( P^{2} \) is equal to \( \sum_{j=0}^{n} |a_{j}|^{2} \). Therefore, had we been able to establish the truth of the middle coefficient conjecture, then the second inequality of Corollary 1 (and hence Theorem 3) would have followed immediately. In other words, the middle coefficient conjecture is equivalent to the conjecture mentioned in the previous section.

B. Suppose \( n \) is even and choose \( \lambda, \ |\lambda|=1 \), so that \( \lambda^{2} = a_{n}/a_{0} \). Then by applying (iii) of Lemma 1 we obtain that for \(|z|=1\), \( \lambda P(z)z^{-n/2} = \text{Re}[Q(z)] \), where \( Q \) is a polynomial of degree \( n/2 \) with leading coefficient, \( 2\lambda a_{n/2} \), and constant coefficient, \( \lambda a_{n/2} \). Therefore if \( |a_{n/2}| \leq 2|a_{n}| \) then \( Q \) has a zero in \(|z| \leq 1| \). It then follows that \( \text{Re}[Q(z)] \) vanishes somewhere on \(|z|=1 \). Thus we have established the following: if \( n \) is even and if \( |a_{n/2}| \leq 2|a_{n}| \) then \( P \) has a zero (and hence at least two) on \(|z|=1 \).

C. If \( n \) is even the polynomials defined by \( \lambda P(z) + (\|P\| + \varepsilon)z^{n/2} \), \( \varepsilon > 0 \), are self-inversive by (iii) of Lemma 1. Clearly they do not vanish on \(|z|=1 \) and so by the previous remark \( |a_{n/2}| + (\|P\| + \varepsilon)| > 2|a_{n}| \). Since \( |a_{n/2}| \leq \|P\| \), \( \varepsilon \) was arbitrary, and \( \lambda a_{n/2} \) is real, it follows that \( |a_{n/2}| + 2|a_{n}| \leq \|P\| \). Therefore we have proved: if \( n \) is even then \( |a_{n/2}| + 2|a_{n}| \leq \|P\| \); in particular the middle coefficient conjecture is true in those cases where \( |a_{n/2}| \leq 2|a_{n}| \).

D. Suppose \( n \) is even and \( P(z) = a_{n} \prod_{j=1}^{n} (z - e^{i\theta}) \). We shall omit the details, but by using the identity

\[
e^{i\theta} - e^{it} = 2i \sin ([\theta - t]/2) e^{i(\theta + t)/2}
\]

we can establish the following: the middle coefficient conjecture is equivalent to the estimate

\[
\left| \frac{1}{\pi} \int_{0}^{2\pi} f(\theta) \, d\theta \right| \leq \max_{\theta} |f(\theta)|,
\]

where \( f \) is the trigonometric polynomial of degree \( n/2 \) defined by \( f(\theta) = \prod_{j=1}^{n} \sin(\theta - \theta_{j})/2 \).

References


Department of Mathematical Sciences, Florida Technological University, Orlando, Florida 32816