FIXED POINT THEOREMS IN UNIFORMLY CONVEX BANACH SPACES

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ABSTRACT. The notion of an asymptotic center is used to prove a number of results concerning the existence of fixed points under certain selfmappings of a closed and bounded convex subset of a uniformly convex Banach space.

1. Introduction. In this paper we shall assume that $X$ is a Banach space with positive modulus of convexity $\delta(\varepsilon)$ (i.e. $X$ is uniformly convex), where

$$\delta(\varepsilon) = \inf\{1 - \frac{1}{2} \|x + y\|: x, y \leq 1, \|x - y\| \geq \varepsilon\}$$

$(0 < \varepsilon \leq 2)$.

Let $\{u_n: n = 1, 2, \cdots\}$ be a bounded sequence in a closed convex subset $C$ of $X$. As in [2] we define

$$(1) \quad r_m(x) = \sup\{\|u_n - x\|: n \geq m\}$$

and denote by $c_m$ the unique point in $C$ with the property that

$$(2) \quad r_m(c_m) = \inf\{r_m(x): x \in C\}.$$

It was shown in [2] that a point $c$, called the asymptotic center of $\{u_n\}$ with respect to $C$, exists such that $c_m \to c$ as $m \to \infty$.

Some basic properties of the asymptotic center are collected in §2 of the present paper. These are then used to obtain a much stronger version of a fixed point theorem proved first in [2]. Next, a fixed point theorem for a countable family of commuting mappings, more general than nonexpansive ones, is proved. A special feature of our results is that, given an orbit of an arbitrary point in $C$, the location of the fixed point is known a priori (as it coincides with the asymptotic center of that orbit) and it has certain uniqueness properties.

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2. Preliminaries.

2.1. We shall make use of the mapping $r$ of $X$ into the reals defined by

$$
(3) \quad r(x) = \inf \{ r_m(x) : m = 1, 2, \cdots \}.
$$

The mapping $r$ as well as the $r_m$ ($m=1, 2, \cdots$) are continuous on all of $X$. This follows easily from the fact that $r_m(x) \to r(x)$ as $m \to \infty$ and

$$
(4) \quad |r_m(x) - r_m(x')| \leq \| x - x' \| \quad (x, x' \in X: m = 1, 2, \cdots).
$$

Also, $r_m(y_m) \to r(y)$ as $y_m \to y$; for

$$
|r_m(y_m) - r(y)| \leq |(r_m(y_m) - r_m(y)) + (r_m(y) - r(y))| \\
\leq \| y_m - y \| + |r_m(y) - r(y)|.
$$

2.2. For the asymptotic center $c$ of \{u_n\} with respect to $C$ we have

$$
(5) \quad x \in C \sim \{c\} \Rightarrow r((x + c)/2) < r(x).
$$

Indeed, since $c_n \to c$ \cite[Theorem 1]{2} there is an $N$ such that, for $n \geq N$, $\| c_n - x \| > \frac{1}{2}\| c - x \|$. By uniform convexity and the definition of $r_n$ and $c_n$ we thus have for $k \geq n \geq N$

$$
\| u_k - (x + c_n)/2 \| \leq r_n(x)(1 - \delta(\| x - c \|/2\rho))
$$

where $\rho$ is a positive constant (e.g. $\rho = r_1(x) + 1$). Thus

$$
r_n((x + c_n)/2) \leq r_n(x)(1 - \delta(\| x - c \|/2\rho))
$$

and therefore

$$
r((x + c)/2) \leq r(x)(1 - \delta(\| x - c \|/2\rho)) < r(x)
$$

as claimed.

2.3. From (5) we conclude easily that

$$
(6) \quad x \in C \sim \{c\} \Rightarrow r(c) < r(x).
$$

For, clearly, $r_m(c_m) \leq r_m(y)$. Thus $r(c) \leq r(y)$ for all $y \in C$. If, however, contrary to (6), $r(c) = r(x)$ then with $y = (x + c)/2$ we would have, by (5), $r(c) > r(y)$.

2.4. If for some $n_0$, $n \geq n_0 \Rightarrow \| u_n - z \| \leq \| u_n - c \|$ then $z = c$. Indeed if $m \geq n_0$ the above implies that $r_m(z) \leq r_m(c)$ and, therefore, $r(z) \leq r(c)$, which, by (6) is only possible if $z = c$.

2.5. In the special case when $X$ is a Hilbert space then $c \in cl \text{co}\{u_n\}$, the closed convex hull of \{u_n\}.

PROOF. If $v \in X$ is not in $cl \text{co}\{u_n\}$ then there is a nearest point $c'$ to $v$ in $cl \text{co}\{u_n\}$. The hyperplane \{ $x \in X : \langle x - c', c' - v \rangle = 0$ \} separates $v$ from $cl \text{co}\{u_n\}$ and we may clearly assume that $\text{Re} \langle u_n - c', c' - v \rangle > 0$,

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(n=1, 2, \cdots). Hence
\[ \|u_n - v\|^2 = \|u_n - c\|^2 + \|c' - v\|^2 + 2 \Re \langle u_n - c', c' - v \rangle \]
\[ > \|u_n - c'\|^2 \quad (n = 1, 2, \cdots). \]

It follows from 2.4 above that v cannot coincide with the asymptotic center c.

3. Fixed point theorems. A well-known theorem due to Browder [1], Göhde [3] and, in a somewhat stronger form, to Kirk [4] states that every nonexpansive mapping of a closed and bounded convex set, in a uniformly convex Banach space, into itself has a fixed point. The next theorem, while assuming substantially less on the mapping, proves the existence of a fixed point having a preassigned location.

**Theorem 1.** Let \( f: C \to C \) be a mapping of a closed convex subset \( C \) of a uniformly convex Banach space into itself and let \( \{ f^n(x) : n=1, 2, \cdots \} \) be a bounded sequence of iterates of some \( x \in C \) having the asymptotic center \( c \) with respect to \( C \). If an \( N \) exists such that
\[ c \|f^n(x) - f(c)\| \leq \|f^{n-1}(x) - c\| \quad (n > N) \]
then \( f(c) = c \).

**Proof.** Let \( c' = f(c) \). Then
\[ \|f^n(x) - c'\| = \|f^n(x) - f(c)\| \leq \|f^{n-1}(x) - c\|, \quad n > N. \]

Hence \( r_n(c') \leq r_{n-1}(c) \) \( (n > N) \) and, therefore, \( r(c') \leq r(c) \). By (6) this can only happen when \( c = c' \), i.e. \( f(c) = c \).

3.1. **Remarks.** (1) The above theorem is stronger than Theorem 2 of [2] in that condition (7) requires less than its counterpart there which requires that the same inequality be satisfied not only for \( c \) but for all points \( v \) of some neighborhood \( V \) of \( c \).

(2) A mapping \( f: C \to C \) has a fixed point if the following condition is satisfied:

For all \( x, y \in C \) there is an \( N = N(x, y) \) such that
\[ \|f^n(x) - f(y)\| \leq \|f^{n-1}(x) - y\| \quad (n > N). \]

Indeed, given an arbitrary \( x \in C \), choose \( y = c \), the asymptotic center of \( \{ f^n(x) \} \). Then \( f(c) = c \).

(Nota that in the corresponding statement in [2] the misprint occurs involving the replacement of \( f(y) \) and \( y \) by \( f^n(y) \) and \( f^{n-1}(y) \) respectively.)

3.2. **If (7) is replaced by**
\[ \|f^n(x) - f^{n-1}(c)\| \leq \|f^{n-1}(x) - c\| \quad (n > N) \]
where \( m \) is a fixed positive integer then the argument used in the proof of Theorem 1 yields the fact that \( f^m(c) = c \) i.e. \( c \) is a periodic point. The proof of the following theorem is based in part on this observation.

**Theorem 2.** Let \( C \) and \( f \) be as in Theorem 1 and suppose that for each \( x \in C \) there is an \( N=N(x) \) and \( m=m(x) \) such that, whenever \( c=c(x) \) is the asymptotic center of \( \{f^n(x)\} \) and \( c \notin \{f^n(x) : n > N\} \),

\[
\|f^n(x) - f^m(c)\| \leq \|f^{n-1}(x) - c\| \quad (n > N)
\]

with strict inequality in the case when \( f^{n-1}(x) \neq c \).

Then, for each \( x \in C \), \( f(c) = c \) (=\( c(x) \)).

**Proof.** As observed before, if \( x \in C \) and \( c = c(x) \), \( f^m(c) = c \). Suppose \( y = c \neq f(c) \) so that \( m > 1 \). Let \( N' = N(y) \), \( m' = m(y) \) and \( c' = c(y) \). Since, again \( f^m(c') = c' \) we obtain from (10) \( \|f^n(y) - c'\| < \|f^{n-1} - c'\| \) \((n > N')\) showing that \( \|f^{N+k}(y) - c'\| \) decreases with increasing \( k \). On the other hand

\[
\|f'^{N+k}(y) - c'\| = \|f'^{N+m}(y) - c'\| < \|f'^{N+m-1}(y) - c'\|
\]

which is impossible. Thus \( c = f(c) \) proving the theorem.

3.3. In the case of nonexpansive mappings the asymptotic center \( c \) of \( \{f^n(x)\} \) for an arbitrary \( x \) in \( C \) is next shown to be the fixed point of \( f \) which is closest to \( \{f^n(x) : n = 0, 1, \ldots\} \).

**Proposition 2.** Let \( f : C \to C \) be a nonexpansive mapping of the closed and bounded convex set \( C \) into itself. If \( x \in C \) and \( c \) is the asymptotic center of \( \{f^n(x) : n = 0, 1, \ldots\} \) then

\[
r(c) = \inf\{\|f^n(x) - c\| : n = 0, 1, \ldots\} \leq \inf\{\|f^n(x) - \xi\| : n = 0, 1, \ldots\}
\]

for each fixed point \( \xi \) of \( f \).

**Proof.** From \( f(\xi) = \xi \) it follows that \( \|f^{n+1}(x) - \xi\| \leq \|f^n(x) - \xi\| \) implying that \( r_n(\xi) = \|f^n(x) - \xi\| \) and \( r(\xi) = \inf\{\|f^n(x) - \xi\| : n = 0, 1, \ldots\} \). The conclusion now follows directly from (6).

4. **Fixed points common to certain families of mappings.** In this section we first prove the existence of a common fixed point to a sequence of commuting mappings satisfying conditions considerably weaker than nonexpansiveness. This is accomplished by producing a sequence of asymptotic centers \( \{c^{(m)}\} \) and showing that its asymptotic center has the desired property. In Hilbert space such a fixed point has a nearest point property analogous to that stated in Proposition 1 for a single mapping. Finally Browder's theorem [1] on the existence of a common fixed point for a family of commuting nonexpansive mappings is shown to hold for the wider class satisfying condition (8).
Theorem 3. Let $C$ be a closed and bounded convex set in a uniformly convex Banach space and $\{f_n:n=0, 1, \cdots\}$ a sequence of commuting mappings of $C$ into itself. Let $c^{(1)}$ be the asymptotic center of $\{f_0^n(x)\}$ for some $x=c(0) \in C$ and, recursively, let $c^{(n)}$ be the asymptotic center of $\{f_{n-1}^m(c^{(n-1)})\}$. Let $c$ be the asymptotic center of $\{c^{(n)}:n=0, 1, \cdots\}$ and suppose that

$$
\|f_k(c^{(l)}) - f_k(c)\| \leq \|c^{(l)} - c\| \quad (l \geq l_k > k; k = 0, 1, \cdots)
$$

(11)

and

$$
\|f_k^n(c^{(l)}) - f_k(c^{(l+1)})\| \leq \|f_k^{n-1}(c^{(l)}) - c^{(l+1)}\| \quad (n \geq N(l, k); l, k = 0, 1, \cdots)
$$

(12)

and

$$
\|f_k f_{l-1}^n(c^{(l-1)}) - f_k(c^{(l)})\| \leq \|f_{l-1}^n(c^{(l-1)}) - c^{(l)}\| \quad (n \geq N(l, k); l, k = 1, 2, \cdots)
$$

(13)

Then $f_k(c) = c$ for all $k=0, 1, \cdots$.

Proof. It suffices to show that for all $l>k$, $k=0, 1, \cdots$

$$
f_k(c^{(l)}) = c^{(l)}.
$$

(14)

Indeed, if this is true then by (11)

$$
\|c^{(l)} - f_k(c)\| = \|f_k(c^{(l)}) - f_k(c)\| \leq \|c^{(l)} - c\|
$$

for $l \geq l_k > k$ and $k=0, 1, \cdots$. By 2.4, then, $f_k(c) = c$. To prove (14) note that $f_k(c^{(k+1)}) = c^{(k+1)}$ by Theorem 1 as a consequence of (12). Assuming then that $f_k(c^{(l-1)}) = c^{(l-1)}$ for $l-1 > k$ we get from (13)

$$
\|f_{l-1}^n(c^{(l-1)}) - f_k(c^{(l)})\| = \|f_k f_{l-1}^n(c^{(l-1)}) - f_k(c^{(l)})\| \\
\leq \|f_{l-1}^n(c^{(l-1)}) - c^{(l)}\| \quad (n \geq N(l, k))
$$

implying, as before, that $f_k(c^{(l)}) = c^{(l)}$.

Proposition 3. Let $X$ be a Hilbert space and $\{f_n:n=0, 1, \cdots\}$ a sequence of commuting nonexpansive mappings of a closed and bounded convex subset $C \subset X$ into itself. Let $\{c^{(n)}\}$ and $c$ be as in Theorem 3. If $\xi$ is any common fixed point of the $f_n$'s then

$$
\inf\{\|c^{(n)} - c\|: n = 0, 1, \cdots\} \leq \inf\{\|c^{(n)} - \xi\|: n = 0, 1, \cdots\}.
$$

Proof. It follows from 2.5 that $\|c^{(n+1)} - \xi\| \leq \|c^{(n)} - \xi\|$, $n=0, 1, \cdots$.

For

$$
\|f_n^k(c^{(n)}) - f_n^k(\xi)\| \leq \|c^{(n)} - \xi\| \quad (n = 0, 1, \cdots; k = 1, 2, \cdots)
$$

and, therefore $c^{(n+1)}$, being in $\text{cl co}\{f_n^k(c^{(n)}): k=1, 2, \cdots\}$, is in the closed
ball $B(\xi, \|e^{(n)} - \xi\|)$. Hence the conclusion of the proposition follows in the same manner as that of Proposition 2.

**Theorem 4.** Let $\{f_a : a \in A\}$ be a family of commuting continuous mappings of a closed and bounded convex set $C$ (in a uniformly convex Banach space $X$) into itself and suppose that for each $a \in A$ and $x, y \in C$ there is an $N(x, y; a)$ such that for $n > N(x, y; a)$

\[(15) \quad \|f_a^{n+1}(x) - f_a(y)\| \leq \|f_a^n(x) - y\|.
\]

Then there is a $\xi \in C$ such that $f_a(\xi) = \xi$ ($a \in A$).

**Proof.** Let $F_a$ be the set of fixed points of $f_a$. We then have to show that $\bigcap \{F_a : a \in A\} \neq \emptyset$. A standard, and often used, argument shows that the commuting property implies that the $F_a$ have the finite intersection property. Thus it suffices to show that each $F_a$ is weakly compact. In view of the reflexivity of $X$ it suffices then to show that each $F_a$ is a nonempty closed and convex subset of $C$. From 3.1(2) we know that $F_a \neq \emptyset$ ($a \in A$). Let then $y, z \in F_a$ and $u = \lambda y + (1 - \lambda)z$ with $0 < \lambda < 1$. Choose $n > \max(N(y, u; a), N(z, u; a))$. Then

\[\|y - f_a(u)\| = \|f_a^{n+1}(y) - f_a(u)\| \leq \|f_a^n(y) - u\| = \|y - u\|
\]

and, similarly, $\|z - f_a(u)\| \leq \|z - u\|$. This, however, in any strictly convex Banach space is only possible when $f_a(u) = u$ i.e. $u \in F_a$. Closedness of each $F_a$ is obvious.

**References**


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