INERTIAL $h$-COBORDISMS WITH FINITE CYCLIC FUNDAMENTAL GROUP

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Abstract. For $M$ a PL $n$-manifold, $n \geq 5$, let $I(M)$ be the subset of torsions $\sigma \in \text{Wh}(\pi_1 M)$ such that the $h$-cobordism $W$ constructed from $M$ with torsion $\sigma$ has its other boundary component PL homeomorphic to $M$. We present three techniques dealing with the determination of $I(M)$ and apply them when $\pi_1 M \cong \mathbb{Z}_q$. We prove: (1) If $n$ is even, $\pi_1 M \cong \mathbb{Z}_q$, $q$ odd, then $I(M) = \text{Wh}(\pi_1 M)$. (2) If $n$ is odd, then there exists $M$ with $\pi_1 M \cong \mathbb{Z}_q$ such that $I(M) = \text{Wh}(\pi_1 M)$.

1. Introduction. We first establish our notation. For convenience we work in the piecewise linear category; similar results hold in the differential and topological categories. $M$ and $N$ will denote closed manifolds of dimension $n \geq 5$. We use $W$ (and $W'$) to denote a compact $(n+1)$-manifold which is an $h$-cobordism between its two boundary components $W_0$, $W_1$. "\(=\)" will denote "is PL homeomorphic to" or "is (group) isomorphic to", depending on the context. We call an $h$-cobordism $W$ inertial if $W_0 \simeq W_1$. $I(M)$ will denote the subset of $\text{Wh}(\pi_1 M)$ consisting of torsions $\sigma$ (we measure our torsions in the domain space except where indicated otherwise) such that the $h$-cobordism $W(M, \sigma)$ with $W_0 = M$ and the torsion of $(W, M)$ (i.e. of $M \subset W$) equal to $\sigma$ is inertial; i.e. $W_1 \simeq M$. Recall that $W(M, \sigma)$ is determined up to a PL homeomorphism which is the identity on $M$ [5].

One of the principal tools of geometric topology is the $s$-cobordism theorem. Frequently it is used in a context where one needs only to know that a particular $h$-cobordism is inertial. Thus a determination of $I(M)$ becomes of interest. In this note we present three techniques which are relevant to this problem and apply them to the case where $\pi_1 M \cong \mathbb{Z}_q$, the finite cyclic group of order $q$. In this case we show that if $q$ is odd and $n$ is even, then $I(M) = \text{Wh}(\pi_1 M)$. If $n$ is odd, the example of lens spaces
shows that \( I(M) \) may be 0 [5]. We show that if \( n \) odd, \( q \) arbitrary, then there is a manifold \( M \) with \( \pi_1 M \cong \mathbb{Z}_q \) such that \( I(M) = \text{Wh}(\pi_1 M) \).

2. Constructions and applications. Our first technique involves an application of surgery theory; for further details on surgery, see [7]. Our terminology will follow [7]. Let \( S_{PL}(M) \) denote the (simple) homotopy triangulations of \( M \). This has a distinguished element \([1_M] = [f]\), where \( f: M' \rightarrow M \) is a simple homotopy equivalence that is homotopic to a PL homeomorphism. Let

\[
H^m(\pi) = \{ \tau \in \text{Wh}(\pi) : \tau = (-1)^m \bar{\tau} \}/\{ \tau + (-1)^m \bar{\tau} \},
\]

where \( \bar{\tau} \) denotes the conjugate of \( \tau \). Rothenburg (cf. [7]) has exhibited an exact sequence, three terms of which are

\[
H^{n+2}(\pi) \xrightarrow{s} L^g_{n+1}(\pi) \xrightarrow{t} L^h_{n+1}(\pi).
\]

Wall [7, p. 108] describes an action of \( L^g_{n+1}(\pi_1 M) \) on \( S_{PL}(M) \), which we denote by \( \cdot \). In particular, the image of \( H^{n+2}(\pi_1 M) \) under \( s \) acts on \([1_M] \in S_{PL}(M)\) as follows. If \( \sigma \in \text{Wh}(\pi_1 M) \) satisfies \( \sigma = (-1)^n \bar{\sigma} \), then \( \sigma \) represents an element \([\sigma] \) of \( H^{n+2}(\pi_1 M) \); \( s([\sigma]) \cdot [1_M] \) is represented by \( ri: W_1 \rightarrow W_0 = M \), where \( W = W(M, \sigma) \), \( i: W_1 \rightarrow W \) is the inclusion and \( r: W \rightarrow W_0 \) is a deformation retraction. The condition \( \sigma = (-1)^n \bar{\sigma} \) is required for \( ri \) to be a simple homotopy equivalence. If \( s([\sigma]) \cdot [1_M] = [1_M] \), then \( ri \) is homotopic to a PL homeomorphism and \( W_1 \cong M \). In general; determining this action is a highly nontrivial problem; however, the difficulties collapse under appropriate algebraic assumptions.

**Proposition 1.** Let \( n \geq 5 \) be even and suppose conjugation is trivial in \( \text{Wh}(\pi_1 M) \). Assume also that the map \( t: L^g_{n+1}(\pi_1 M) \rightarrow L^h_{n+1}(\pi_1 M) \) is injective. Then \( I(M) = \text{Wh}(\pi_1 M) \).

**Proof.** Since \( n \) is even and conjugation is trivial, each \( \sigma \in \text{Wh}(\pi_1 M) \) satisfies \( \sigma = (-1)^n \bar{\sigma} \). If \( s([\sigma]) \cdot [1_M] = [1_M] \), then \( \sigma \in I(M) \). But im \( s = \ker t = 0 \).

**Corollary 1.** If \( n \geq 5 \) is even and \( \pi_1 M \cong \mathbb{Z}_q \), \( q \) odd, then \( I(M) = \text{Wh}(\pi_1 M) \).

**Proof.** Conjugation is trivial in \( \text{Wh}(\mathbb{Z}_q) \) [2]. A. Bak [1] has shown that \( L^g_{n+1}(\pi) = 0 = L^h_{n+1}(\pi) \) for \( \pi \) a finite abelian group of odd order, hence for \( \pi = \mathbb{Z}_q \).

**Remarks.** 1. The ideas behind Proposition 1 have been known for some time. What was needed to apply them were computations of Wall groups such as Bak's.
2. Note that Corollary 1 is nontrivial since Wh(\(Z_q\)) is a direct sum of \(f(q)/2-1\) copies of \(Z\), \(q \geq 3\), where \(f\) is the Euler \(\phi\)-function [2].

3. \(2\text{Wh}(\pi_1 M) = \{ \sigma \in \text{Wh}(\pi_1 M) : \sigma = 2\tau \}\) is easily seen to lie in \(I(M)\) via the doubling construction of Milnor [5] when conjugation is trivial. In this case \(H^{n+2}(\pi_1 M) = \text{Wh}(\pi_1 M)/2\text{Wh}(\pi_1 M)\).

4. Let \(\pi_1 M\) be a finite abelian group of odd order, \(C = \{ \sigma \in \text{Wh}(\pi_1 M) : \sigma = \bar{\sigma} \}\). Then the proof of Proposition 1 shows that \(C \subset I(M)\).

When \(n\) is odd, the technique above appears to be more difficult to apply; it is useless for \(\pi_1 M \cong Z_q\). A relevant question for \(n\) odd is one of realizability: Given \(\sigma \in \text{Wh}(\pi)\), does there exist a manifold \(M\) with \(\pi_1 M = \pi\) and \(\sigma \in I(M)\)? In particular, can one find \(M\) such that \(\text{Wh}(\pi_1 M) = I(M)\)? The Milnor doubling construction yields no examples when \(\sigma = \bar{\sigma}\), as in \(\text{Wh}(Z_q)\). However, a minor modification of it (which is also related to Farrell’s fibering theorem [3]) does lead to some results.

Suppose \(W\) is an \(h\)-cobordism with \(W_0 = M, W_1 = N\) and \(W’\) is an \(h\)-cobordism with \(W’_0 = N\) and \(W’_1 \simeq M\). Suppose \(f : W \rightarrow N\) is a PL homeomorphism. Let \(\sigma, \sigma’ \in \text{Wh}(\pi_1 N)\) be the torsions of \((W, M)\) and \((W’, N)\) measured in \(\text{Wh}(\pi_1 W)\) and \(\text{Wh}(\pi_1 W’)\), respectively, and \(r_N : W \rightarrow N, r’_N : W’ \rightarrow N\) are deformation retractions, then \((r_N)_{\#} \sigma_1 = \sigma\) and \((r’_N)_{\#} \sigma’ = \sigma’\); it turns out to be technically simpler to work in \(\text{Wh}(\pi_1 N)\). Then the torsion of \((W \cup _f W’, M)\) as measured in \(\text{Wh}(\pi_1 N)\) is \(\sigma + f_{\#} \sigma’\).

Two special cases of interest are where \(W’ = W\) (inverted so \(W_0’ = W_1, W_1’ = W_0\)) and where \(W’\) is the inverse of \(W\), i.e. \(W \cup W’ \simeq M \times I\). In the first case \(\sigma’ = -\bar{\sigma}\); in the second case \(\sigma’ = -\sigma\). Thus \((r_M i_N)_{\#} (\sigma - f_{\#} \bar{\sigma})\) and \((r_M i_N)_{\#} (\sigma - f_{\#} \sigma)\) are in \(I(M)\) where \(i_N : N \rightarrow W\) is the inclusion and \(r_M : W \rightarrow M\) is a deformation retraction. If \(\sigma = \bar{\sigma}\), the two cases coincide.

The first case is merely a slight modification of the doubling construction. The second case arises naturally in trying to fiber \(N_f = N \times [0, 1]/(x, 1) \sim (f(x), 0)\) over the circle when one splits along \(M\). For this construction to be of any value, there must exist manifolds \(N\) and PL homeomorphisms \(f\) of \(N\) which induce nontrivial maps \(f_*\) of \(\pi_1 N\) and then \(f_{\#}\) of \(\text{Wh}(\pi_1 N)\).

Our next proposition concerns \(f_*\) when \(\pi_1 N = Z_q\); we only examine \(f_{\#}\) for the special case of \(Z_q\).

**Proposition 2.** Let \(\alpha_r\) be the automorphism of \(Z_q\) such that \(\alpha_r(1) = r, (r, q) = 1\). Then there is a manifold \(N\) (of any given dimension \(\geq 5\)) and a PL homeomorphism \(f : N \simeq\) with \(\pi_1 N = Z_q\) and \(f_* = \alpha_r\).

**Proof.** Let \(P_q\) denote the pseudoprojective plane \(S^1 \cup q e^2\). Olum [6] proves that there is a simple homotopy equivalence \(g : P_q \simeq\) with \(g_\# = \alpha_r\) (under the natural isomorphism \(\pi_1 P_q \simeq Z_q\)). Embed \(P_q\) as a subcomplex of \(E^{n+1}\), \(n \geq 6\) and let \(N\) be the boundary of a regular neighborhood \(R\) of \(P_q\).
The composition of $g$ with the inclusion of $P_\varphi$ into the interior of $R$ is homotopic to an imbedding $h$. By [4] and uniqueness of regular neighborhoods, $h$ may be extended to an imbedding $k$ of $R$ into the interior of $R$. Since $g$ is a simple homotopy equivalence, so is $k$; excision and the $s$-cobordism theorem then imply that $R \setminus \text{int } k(R) \simeq N \times [0, 1]$. Using collars, we may modify $k$ to give a PL homeomorphism $l: R \to R$. Letting $f= l|_N$, then $f_\ast = \alpha_r$ (in terms of the isomorphisms $\pi_1 N \simeq \pi_1 R \simeq \pi_1 P_\varphi \simeq \mathbb{Z}_q$).

Thus we may realize any possible automorphism of $\mathbb{Z}_q$. The next question is whether $(\alpha_r)_\# : \text{Wh}(\mathbb{Z}_q) \cong \ker I(N)$ is ever nontrivial. Since we have a more effective technique for $\mathbb{Z}_q$, we content ourselves with one example: $q=5$. Then $\text{Wh}(\mathbb{Z}_5) \cong \mathbb{Z}$, generated by the unit $t + t^{-1} - 1 \in \mathbb{Z}(\mathbb{Z}_5)$, with inverse $t^2 + t^{-2} - 1$, where $t$ denotes the generator of $\mathbb{Z}_5$ [5]. Thus the automorphism $\alpha_2 : \mathbb{Z}_5 \cong \ker I(M)$, where $M= W_1(N, [t + t^{-1} - 1])$, $N$ as in Proposition 2.

The proof of Proposition 2 suggests another technique which yields our realizability theorem for $\mathbb{Z}_q$ using another result of Olum [6].

**Proposition 3.** Let $n \geq 5$, $X \subseteq E^{n+1}$ a 2-complex and $g : X \hookrightarrow$ a homotopy equivalence with torsion $\tau \in \text{Wh}(\pi_1 X)$. Then if $N$ is the boundary of a regular neighborhood $R$ of $X$, we have $F_\# \tau \in I(N)$ where $F$ is the composition of isomorphisms $\pi_1 X \cong \pi_1 R \cong \pi_1 N$ induced by inclusions.

**Proof.** Homotope $X \hookrightarrow Y \subseteq \text{int } R$ to an imbedding $h$. $h$ then extends to an imbedding $k : R \to \text{int } R$ by [4]. Then $R \setminus \text{int } k(R)$ is an inertial h-cobordism with boundary $N \cup k(N)$. Using excision, one computes the torsion of $k(N) \subset R \setminus \text{int } k(N))$ where $G : \pi_1 X \to \pi_1 k(N)$ is the composition $\pi_1 X \to * \pi_1 X \to * \pi_1 k(R) \to * \pi_1 k(N)$. Transferring from $k(N)$ to $N$ via $(k|N)^{-1}$ gives $F_\# \tau \in I(N)$ for $F = (k|N)^{-1} G$. One then checks $F$ is just the composition of isomorphisms $\pi_1 X \cong \pi_1 R \cong \pi_1 N$ induced by inclusions.

**Corollary 2.** Given $n \geq 5$, $q \in \mathbb{N}$, there is a manifold $N$ of dimension $n$ with $\pi_1 N \simeq \mathbb{Z}_q$ and $\text{Wh}(\pi_1 N) = I(N)$.

**Proof.** This follows from Proposition 3 with $X= P_\varphi$ using Olum’s result [6] that there are self homotopy equivalences of $P_\varphi$ with prescribed torsion in $\text{Wh}(\pi_1 P_\varphi)$.

**Remark.** Propositions 2 and 3 make it clear that the realizability problem is really one of CW complexes and (simple) homotopy equivalences.

**References**


