MAXIMAL SUBLATTICES OF FINITE DISTRIBUTIVE LATTICES. II

IVAN RIVAL

Abstract. Let \( L \) be a lattice, \( J(L) = \{ x \in L | x \text{ join-irreducible in } L \} \) and \( M(L) = \{ x \in L | x \text{ meet-irreducible in } L \} \). As is well known, the sets \( J(L) \) and \( M(L) \) play a central role in the arithmetic of a lattice \( L \) of finite length and particularly, in the case that \( L \) is distributive. It is shown that the "quotient set" \( Q(L) = \{ b | a, b \in J(L), a \leq b \} \) plays a somewhat analogous role in the study of the sublattices of a lattice \( L \) of finite length. If \( L \) is a finite distributive lattice, its quotient set \( Q(L) \) in a natural way determines the lattice of all sublattices of \( L \). By examining the connection between \( J(K) \) and \( J(L) \), where \( K \) is a maximal proper sublattice of a finite distributive lattice \( L \), the following is proven: every finite distributive lattice of order \( n \geq 3 \) which contains a maximal proper sublattice of order \( m \) also contains sublattices of orders \( n - m, 2(n-m) \), and \( 3(n-m) \); and, every finite distributive lattice \( L \) contains a maximal proper sublattice \( K \) such that either \( |K| = |L| - 1 \) or \( |K| \geq 2l(L) \), where \( l(L) \) denotes the length of \( L \).

1. Introduction. Let \( L \) be a lattice, \( J(L) = \{ x \in L | x \text{ join-irreducible in } L \} \) and \( M(L) = \{ x \in L | x \text{ meet-irreducible in } L \} \). As is well known the sets \( J(L) \) and \( M(L) \) play a central role in the arithmetic of a lattice \( L \) of finite length and particularly, in the case that \( L \) is distributive. We show (Proposition 1) that the "quotient set" \( Q(L) = \{ b | a, b \in J(L), a \leq b \} \) plays a somewhat analogous role in the study of the sublattices of a lattice \( L \) of finite length. If \( L \) is a finite distributive lattice, its quotient set \( Q(L) \) in a natural way determines (Theorem 1) the lattice \( \text{Sub}(L) \) of all sublattices of \( L \).

By examining (Theorem 2) the connection between \( J(K) \) and \( J(L) \), where \( K \) is a maximal proper sublattice of a finite distributive lattice \( L \), we can derive some useful information about the orders of sublattices of finite distributive lattices; namely, every finite distributive lattice of order \( n \geq 3 \) which contains a maximal proper sublattice of order \( m \) also contains
sublattices of orders \( n-m, 2(n-m), \) and \( 3(n-m) \); and, every finite distributive lattice \( L \) contains a maximal proper sublattice \( K \) such that either \( |K|=|L|-1 \) or \( |K|\geq 2l(L) \), where \( l(L) \) denotes the length of \( L \).

The author wishes to thank Barry Wolk for suggesting the proof presented here for Proposition 1. For all terminology not explained here we refer to G. Birkhoff [1].

2. A connection between \( Q(L) \) and \( \text{Sub}(L) \). Proposition 1 below serves to underline a basic connection between \( Q(L) \) and the sublattices of a lattice \( L \) of finite length, a connection which, specialized to finite distributive lattices, has been the motivation for the results presented in this paper.

Proposition 1, in fact, is a generalization of Lemma 1 [2]. We shall throughout adopt the abbreviation \( \bigcup_{\mathcal{A}} [a, b] \) for \( \bigcup_{b/a\in \mathcal{A}} [a, b] \), where \( A \subseteq Q(L) \).

**Proposition 1.** If \( S \) is a sublattice of a lattice \( L \) of finite length then \( S=L-\bigcup_{\mathcal{A}} [a, b] \), for some \( A \subseteq Q(L) \).

**Proof.** We must show that for every \( x \in L-S \) there is some \( b/a \in Q(L) \) such that \( x \in [a, b] \subseteq L-S \). Let us suppose that this does not hold for some \( x \in L-S \). Let \( A=\{a \in J(L)\mid a \leq x \} \) and \( B=\{b \in M(L)\mid x \leq b \}; \) clearly, \( A \neq \emptyset \neq B \) and \( \bigvee A = x = \bigwedge B \). But then by our assumption, for every \( a \in A \) and for every \( b \in B \) there exists \( y_{ba}^b \in S \cap [a, b] \). Since \( L \) is of finite length it is complete; therefore, \( \bigvee_{a \in A} \bigwedge_{b \in B} y_{ba}^b = x \in L-S \), which is a contradiction.

In view of Proposition 1 it is natural to classify sublattices of a lattice \( L \) of finite length in terms of subsets of \( Q(L) \). Indeed, for \( A \subseteq Q(L) \) we define \( \text{Cl}(A)=\{y/x \in Q(L)\mid [x, y] \subseteq \bigcup_{\mathcal{A}} [a, b] \} \) and \( \text{Cl}(Q(L))=\{\text{Cl}(A)\mid A \subseteq Q(L) \} \).

The following lemma is straightforward.

**Lemma 1.** Let \( L \) be a lattice of finite length and \( A, B \subseteq Q(L) \). Then
(i) \( \bigcup_{\mathcal{A}} [a, b] = \bigcup_{\text{Cl}(A)} [x, y] \) and,
(ii) \( \bigcup_{\text{Cl}(A)} [x, y] \subseteq \bigcup_{\text{Cl}(B)} [u, v] \) if and only if \( \text{Cl}(A) \subseteq \text{Cl}(B) \).

The next lemma is an easy consequence of Lemma 1.

**Lemma 2.** Let \( L \) be a lattice of finite length. Then
(i) \( \text{Cl} \) is a closure operator on \( Q(L) \) and,
(ii) \( \text{Cl}(Q(L)) \) is a lattice with respect to set inclusion.
Theorem 1. For a lattice \( L \) of finite length the following conditions are equivalent:

(i) \( L \) is distributive;

(ii) \( L - \bigcup A [a, b] \) is a sublattice of \( L \) for every \( A \subseteq Q(L) \);

(iii) for every \( S \subseteq L \), \( S \) is a sublattice of \( L \) if and only if \( S = L - \bigcup A [a, b] \) for some \( A \subseteq Q(L) \);

(iv) the mapping \( \varphi(S) = \text{Cl}(A) \), where \( S = L - \bigcup A [a, b] \), \( A \subseteq Q(L) \), is an isomorphism between \( \text{Sub}(L) \) and the dual of \( \text{Cl}(Q(L)) \).

Proof. That (i) implies (ii) follows from the fact that join-irreducible elements in a distributive lattice are join-prime, that is, if \( a \in J(L) \) and \( a \leq b \lor c \) then \( a \leq b \) or \( a \leq c \). Applying Proposition 1 we get that (ii) implies (iii). On the other hand, Proposition 1 together with Lemma 1(ii) shows that \( \varphi \) is well-defined, one-one, isotone, and that, in fact, \( \varphi^{-1} \) is isotone. From (iii) we have that \( \varphi \) is onto, so that \( \varphi \) is, indeed, an isomorphism; thus, (iii) implies (iv). It remains only to show that (iv) implies (i).

Let \( M_5 \) and \( N_5 \) be the two five-element nondistributive lattices labelled as in Figure 1. Suppose that \( F \) satisfies (iv) but \( F \) is nondistributive. Then \( L \) contains as a sublattice a copy of \( M_5 \) or \( N_5 \). Let \( d \) be a join-irreducible in \( L \) such that \( d \leq a \) but \( d \not\leq b \land c \), and \( e \) a meet-irreducible in \( L \) such that \( e \leq b \lor c \). By the surjectivity of \( \varphi^{-1} \), \( L - \bigcup \text{Cl}(\{e/d\}) [x, y] \) is a sublattice of \( L \).

In view of Lemma 1(i), \( L - \bigcup \text{Cl}(\{e/d\}) [x, y] = L - [d, e] \). But \( b \lor c \in [d, e] \) although \( b, c \in L - [d, e] \), which is a contradiction. Thus, (iv) implies (i), completing the proof.

3. Maximal proper sublattices of finite distributive lattices. We define a partial ordering on \( Q(L) \) as follows: \( b/a \leq d/c \) if and only if \( [a, b] \subseteq [c, d] \).

If \( b/a \) is minimal with respect to this ordering then \( \text{Cl}(\{b/a\}) = \{b/a\} \) so that by Theorem 1, \( L - [a, b] \) is a maximal proper sublattice of \( L \) in the case that \( L \) is finite distributive. Note that if \( b/a \in Q(L) \) then \( b/a \) is minimal if and only if \( [a, b] \subseteq L - M(L) \) and \( (a, b) \subseteq L - J(L) \) (cf. [2, Theorem 3]).
For $x, y \in L$, $x$ covers $y$ ($x \triangleright y$ or $y \triangleleft x$) in $L$ if $x \triangleright y$ and $x \triangleright z \triangleright y$ implies $x = z$, for every $z \in L$. For $A \subseteq L$ we define $\text{cov}(A) = \{x \in L \mid x \triangleright a$ or $x \triangleleft a$ or $x = a$, for some $a \in A\}$. Observe that $a \in L - J(L)$ ($a \in L - M(L)$) if and only if there exist $b, c \in \text{cov}(\{a\})$ such that $a = b \lor c$ ($a = b \land c$).

**Theorem 2.** Let $L$ be a finite distributive lattice and $K = L - [a, b]$ ($b/a \in Q(L)$, $a \neq b$) be a maximal proper sublattice of $L$. Then (i) $\text{cov}([a, b])$ is a sublattice of $L$ isomorphic to the direct product of $[a, b]$ with a three-element chain, and (ii) $J(K) = (J(L) - \{a\}) \cup \{c\}$, where $a < c \in K$.

**Proof.** Set

\[
A = \{y \in K \mid y < x$ for some $x \in [a, b]\},
\]

\[
B = \{y \in K \mid y > x$ for some $x \in [a, b]\},
\]

\[
A' = \{x \in [a, b] \mid x > y$ for some $y \in A\},
\]

\[
B' = \{x \in [a, b] \mid x < y$ for some $y \in B\}.
\]

To establish (i) it suffices to show that $A \cong [a, b] \cong B$. Since $b/a$ is minimal in $Q(L)$ and $a \neq b$, it follows that $a \neq 0$ and $b \neq 1$; thus, $a \in A'$ and $b \in B'$. Furthermore, since $L - [a, b]$ is a sublattice of $L$, every element in $A'$ covers precisely one element in $A$ and every element in $B'$ is covered by precisely one element in $B$.

Suppose now that $c_1', c_2'$ are distinct minimal elements in $B'$ with covers $c_1, c_2 \in B$. Since $[a, b]$ is a sublattice of $L$, $c_1 \neq c_2$; since $c_1'$ is incomparable with $c_2'$, $c_1$ is incomparable with $c_2$; and since $L - [a, b]$ is a sublattice of $L$, $c_1, c_2 \geq c_1 \land c_2 \in L - [a, b]$. Now, if $c_2' = c_2' \lor (c_1 \land c_2)$ then $c_1 \land c_2' \leq c_1 \land c_2 < c_2'$, so that $c_1 \land c_2 \in [a, b]$. Therefore, $c_1 \land c_2 \leq c_2'$ and, since $c_2' < c_2$, we have that $c_1 \land c_2 \leq (c_1 \land c_2) = c_2$ which, by transposition implies that $c_2' \land (c_1 \land c_2) < c_1 \land c_2$. But $c_1 \land c_2 \leq c_2' \land (c_1 \land c_2) < c_2'$ so that $c_2' \land (c_1 \land c_2) < c'$, contradicting the minimality of $c_2'$. Thus, $B'$ has a unique minimal element $c'$ with precisely one cover $c$ in $B$; dually, $A'$ has a unique maximal element $d'$ covering precisely one element $d$ in $A$. Now, if $f$ is the unique cover of $b$ and $e$ the unique element covered by $a$ then by transposition we have that $A = [e, d] \cong [a, d'] = A'$ and $B = [c, f] \cong [c', b] = B'$. From this it follows that $\text{cov}([a, b]) = A \cup [a, b] \cup B$ is a sublattice of $L$ and that in fact, $b/a$ is minimal in $Q(\text{cov}([a, b]))$. In this case $A \cup B$ is a maximal proper sublattice of $\text{cov}([a, b])$ so that by [2, Theorem 2], $|A \cup B| \geq \frac{3}{2} \text{cov}([a, b])$. Now, if $d < b$ or $a < c$ then $|\text{cov}([a, b])| = |A| + |[a, b]| + |B| < 3|a, b|$. But $[a, b] = \text{cov}([a, b]) - (A \cup B)$ so that $|A \cup B| < \frac{3}{2} \text{cov}([a, b])$, which is a contradiction. Thus, $a = c'$ and $b = d'$ so that $A \cong [a, b] \cong B$, from which (i) follows.

To show (ii) observe first that $J(L) - \{a\} \subseteq J(K)$ and $J(K) \cap A \subseteq J(L) - \{a\}$. It suffices then to show that $J(K) \cap B = \{c\}$.
Let \( x \in B - \{c\} \). Choose some \( y \in B \) such that \( x > y \). Then there exist \( x_1, y_1 \in [a, b] \) and \( x_2, y_2 \in A \) such that \( x > x_1 > x_2 \) and \( y > y_1 > y_2 \). By transposition \( x_1 > y_1, x_2 > y_2, \) and \( x_1 \wedge y = y_1 \). If \( x_2 < y \) then \( y_1 = x_1 \wedge y \geq x_2 > y_2, \) and since \( y_1 > y_2 \) we have that \( y_1 = x_2 \), which is impossible. Thus, \( x_2 \) is incomparable with \( y \) and, in fact, \( x \) covers \( x_2 \) in \( K \), and since \( x \) also covers \( y \) in \( K \), we get that \( x \) is join-reducible in \( K \).

It remains only to show that \( c \in J(K) \). We may without loss of generality assume that \( c \) covers two distinct incomparable elements \( c_1, c_2 \in L \), both incomparable with \( a \). But \( a \) is join-irreducible in \( L \), that is, it covers only \( e \). By transposition we get that \( \{a, c_1, c_2, e, c\} \) is a sublattice of \( L \) isomorphic to the five-element modular, nondistributive lattice \( M_5 \) which, of course, is a contradiction. The proof of the theorem is now complete.

The following corollary is an immediate consequence of Theorem 2(i).

**Corollary 1.** Every distributive lattice of order \( n \geq 3 \) which contains a maximal proper sublattice of order \( m \) also contains sublattices of orders \( n - m, 2(n - m), \) and \( 3(n - m) \).

**Corollary 2.** Every finite distributive lattice \( L \) contains a maximal proper sublattice \( K \) such that either \( |K| = |L| - 1 \) or \( |K| \leq 2l(L) \).

**Proof.** We may without loss of generality assume that \( \text{Irr}(L) = \emptyset \). Recall that for finite distributive lattices \( |J(L)| = l(L) + 1 = |M(L)| \). Furthermore, the inequality \( |L| \geq |J(L)| + |M(L)| - |\text{Irr}(L)| \) holds in every lattice \( L \) of finite length, so that if \( L \) is distributive we have that \( |L| \geq 2(l(L) + 1) - |\text{Irr}(L)| \). (This latter inequality, incidentally, holds in every lattice of finite length, cf. [3, Theorem 1].

If \( J(K) = J(L) - \{a\} \) then \( M(K) = M(L) - \{b\} \), and since \( J(L) \cap M(L) = \text{Irr}(L) = \emptyset \) we also have that \( \text{Irr}(K) = \emptyset \). In this case \( |K| \geq 2l(L) - |\text{Irr}(K)| = 2l(L) \).

Otherwise, \( J(K) \neq J(L) - \{a\} \). By Theorem 2(ii) and its dual there exist \( c, d \in L \) such that \( J(K) = (J(L) - \{a\}) \cup \{c\}, c \notin J(L), \) and \( M(K) = (M(L) - \{b\}) \cup \{d\}, d \notin M(L) \). Observe that \( (J(L) - \{a\}) \cap (M(L) - \{b\}) \subseteq J(L) \cap M(L) = \text{Irr}(L) = \emptyset \). Therefore, \( \text{Irr}(K) \subseteq \{c, d\} \), so that in this case

\[
|K| \geq 2l(K) - |\text{Irr}(K)|
\]

\[
\geq 2(|J(L) - \{a\}) \cup \{c\}| + 1 - 2 = 2l(L).
\]

The estimate on the order of maximal proper sublattices of finite distributive lattices prescribed in Corollary 2 is best possible in the sense that, if for every positive integer \( n \), \( L_n \) is the ordinal sum of \( n \) copies of the Boolean lattice \( 2^3 \), then the maximum order of a maximal proper sublattice of \( L_n \) is \( 2l(L_n) \).
REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANITOBA, WINNIPEG, MANITOBA R3T 2N2, CANADA