THE APPROXIMATION OF ONE-ONE MEASURABLE
TRANSFORMATIONS BY MEASURE PRESERVING
HOMEOMORPHISMS

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Abstract. This paper contains two results related to the material in [2]. Suppose \( f \) is a one-one transformation of the open unit interval \( I^n \) (where \( n \geq 2 \)) onto \( I^n \). 1. If \( f \) is absolutely measurable and \( \varepsilon > 0 \), then there is an absolutely measurable homeomorphism \( \varphi_\varepsilon \) of \( I^n \) onto \( I^n \) such that \( m(\{x: f(x) \neq \varphi_\varepsilon(x) \}) < \varepsilon \), where \( m \) denotes \( n \)-dimensional Lebesgue measure. 2. Suppose \( \mu \) is either (1) a nonatomic, finite Borel measure on \( I^n \) such that \( \mu(G) > 0 \) for every nonempty open subset \( G \) of \( I^n \), or (2) the completion of a measure of type (1). If \( f \) is \( \mu \)-measure preserving and \( \varepsilon > 0 \), then there is a \( \mu \)-measure preserving homeomorphism \( \varphi_\varepsilon \) of \( I^n \) onto \( I^n \) such that \( \mu(\{x: f(x) \neq \varphi_\varepsilon(x)\}) < \varepsilon \).

1. For any subset \( S \) of \( n \)-dimensional Euclidean space \( \mathbb{R}^n \), denote by \( \mathcal{M}(S) \) the set of all measures \( \mu \) such that \( \mu \) is either (1) a nonatomic, finite, Borel measure on \( S \) such that \( \mu(G) > 0 \) for every nonempty open subset \( G \) of \( S \), or (2) the completion of a measure of type (1). If, for \( i = 1, 2, S_i \) is a subset of \( R^n \) and \( \mu_i \in \mathcal{M}(S_i) \), and \( f \) is a one-one transformation of \( S_1 \) onto \( S_2 \), then we say that \( f \) carries \( \mu_1 \) into \( \mu_2 \) provided \( f(D(\mu_1)) = D(\mu_2) \) and \( \mu_2(f[A]) = \mu_1(A) \) for every \( A \) in \( D(\mu_1) \), where \( D(\mu_i) \) is the domain of \( \mu_i \). If \( S_1 = S_2 \) and \( \mu_1 = \mu_2 \), then we say that \( f \) is \( \mu \)-measure preserving.

In this note, we show how a minor modification of the proof of Theorem 5 of [2] yields the following result.

Theorem 1. Suppose \( \mu_1, \mu_2 \in \mathcal{M}(I^n) \), where \( n \geq 2 \) and \( I^n \) denotes the open unit interval in \( \mathbb{R}^n \), and \( f \) is a one-one transformation of \( I^n \) onto \( I^n \) which carries \( \mu_1 \) into \( \mu_2 \). For every \( \varepsilon > 0 \), there is a homeomorphism \( \varphi_\varepsilon \) of \( I^n \) onto \( I^n \) which carries \( \mu_1 \) into \( \mu_2 \) such that

\[
\mu_2(\{x: f(x) \neq \varphi_\varepsilon(x)\}) = \mu_1(\{x: f^{-1}(x) \neq \varphi_\varepsilon^{-1}(x)\}) < \varepsilon.
\]

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REMARKS. The author has been informed recently by J. C. Oxtoby that he has, in his paper *Approximation by measure-preserving homeomorphisms*, generalized Theorem 1 (with $\mu_1=\mu_2$). In doing so, he re-proved this statement. His work was done independently and was done after the work in this paper.

A one-one transformation $f$ of $I^n$ onto $I^n$ is called absolutely measurable [2] if $f[A]$ and $f^{-1}[A]$ are Lebesgue measurable for every Lebesgue measurable subset $A$ of $I^n$.

We then obtain the following result as a corollary to Theorem 1.

**Theorem 2.** If $f$ is an absolutely measurable, one-one transformation of $I^n$ onto $I^n$ (where $n \geq 2$) and $\varepsilon > 0$, then there is an absolutely measurable homeomorphism $\varphi_\varepsilon$ of $I^n$ onto $I^n$ such that

$$m(\{x : f(x) \neq \varphi_\varepsilon(x) \text{ or } f^{-1}(x) \neq \varphi_\varepsilon^{-1}(x)\}) < \varepsilon,$$

where $m$ denotes $n$-dimensional Lebesgue measure.

2. In this section $n$ will always denote a fixed integer $\geq 2$. By an $(n-1)$-dimensional interval in $R^n$ we mean a set of the form

$$\{(x_1, \cdots, x_n) \in R^n : x_k = c \} \cap \prod \{(a_j, b_j) : j = 1, \cdots, n\},$$

where $k$ is an integer such that $1 \leq k \leq n$, $c$ is a real number, and, for $j=1, \cdots, n$, $a_j$ and $b_j$ are real numbers such that $a_j < b_j$. For any subset $A$ of $R^n$, we denote the interior of $A$, the closure of $A$, and the boundary of $A$ by int $A$, cl $A$, and bdry $A$, respectively.

**Definition.** A subset $P$ of $R^n$ is called a $p$-set if $P$ is a combinatorial $n$-ball (see p. 18 of [1]) and bdry $P$ is the union of a finite number of $(n-1)$-dimensional intervals.

**REMARKS.** (1) The $p$-sets used in the proof of Theorem 1 (and Theorem 5 of [2]) can be chosen to be very simple "snake-like" objects.

(2) The author wishes to thank Dr. L. C. Glaser for answering a number of questions concerning Lemma 5 of [2].

The following statement follows from Corollary 3 of [3] and Lemma 5 of [2].

**Lemma 1.** Suppose, for $i=1, 2$, that \{P(i), j=1, \cdots, r\} is a disjoint family of $p$-sets contained in the interior of the $p$-set $P(i)$. For $i=1, 2$, let $Q(i) = P(i) \cap \bigcup \{\text{int } P(i, j) : j=1, \cdots, r\}$, and suppose $\mu_1 \in \mathcal{M}(Q(i))$ and $\mu_2(\text{bdry } Q(i))=0$. If $\mu_1(Q(1))=\mu_2(Q(2))$, then every homeomorphism $\varphi$ of $\text{bdry } P(1)$ onto $\text{bdry } P(2)$ can be extended to a homeomorphism $\varphi^*$ of $Q(1)$ onto $Q(2)$ which carries $\mu_1$ into $\mu_2$ such that $\varphi^*[\text{bdry } P(1, j)]=\text{bdry } P(2, j)$ for $j=1, \cdots, r$. 

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The following statement follows easily from the definition of sectionally zero dimensional set [2, p. 263].

**Lemma 2.** Suppose $K$ is a sectionally zero dimensional, compact set contained in the interior of the $p$-set $P$ such that $m(K) < \gamma < m(P)$. Then there is a $p$-set $Q$ such that $K \subset \text{int } Q$, $Q \subset \text{int } P$, and $m(Q) = \gamma$.

**Lemma 3.** Suppose $P$, $Q$ are $p$-sets contained in $I^n$, and $S$ and $T$ are compact, sectionally zero dimensional sets contained in $\text{int } P$ and $\text{int } Q$, respectively. If $\varphi$ is an $m$-measure preserving homeomorphism of $S$ onto $T$ and $m(P) = m(Q)$, then $\varphi$ can be extended to an $m$-measure preserving homeomorphism of $P$ onto $Q$.

We obtain a proof of Lemma 3 by making the following modifications in the proof of Theorem 1 of [2]. At the kth step of the definition of the auxiliary sets, since $m(S_{j_1,\ldots,j_k}) = m(T_{j_1,\ldots,j_k})$ for $j_1 \leq m_1, \ldots, j_k \leq m_1, \ldots, j_k$, by Lemma 2, the $p$-sets $P_{j_1,\ldots,j_k}$, $Q_{j_1,\ldots,j_k}$ can be chosen so that $m(P_{j_1,\ldots,j_k}) = m(Q_{j_1,\ldots,j_k})$ for $j_1 \leq m_1, \ldots, j_k \leq m_1, \ldots, j_k$. Then, at the kth step in defining the extension of $\varphi$, instead of Lemma 5 of [2], we use Lemma 1.

**Remark.** In proving Theorem 1 of [2], C. Goffman uses Lemma 4 of [2]. Lemma 4 of [2] is false. However, if the following sentence is added to the hypothesis of Lemma 4, then the resulting lemma is true. For each $i$, there is an interval $J_i$ such that $F_i \subset \text{int } J_i$ and $J_i \subset P$. The modified version of Lemma 4 of [2] is sufficient for the proof of Lemma 3 (and Theorem 1 of [2]).

**Proof of Theorem 1.** If $\mu_1 = \mu_2 = m$, Theorem 1 follows from Lemma 3 in exactly the same way as Theorem 5 of [2] follows from Theorem 1 of [2]. Now, suppose $\mu_1$, $\mu_2$ are arbitrary elements of $M(I^n)$ and $f$ is as hypothesized. First, note that either both $\mu_1$ and $\mu_2$ are of type (1) or both $\mu_1$ and $\mu_2$ are of type (2). Hence, we may assume that both $\mu_1$ and $\mu_2$ are of type (2) and that $\mu_1(I^n) = 1$. By Theorem 2 of [3], there are homeomorphisms $\psi$ and $\varphi$ of $cI^n$ onto $cI^n$ such that $\psi$ carries $m$ into $\mu_1$ and $\varphi$ carries $\mu_2$ to $m$. Then $f^* = \varphi \circ f \circ \psi$ is $m$-measure preserving. If $\theta$ is an $m$-measure preserving homeomorphism of $I^n$ onto $I^n$ such that $m(\{x : f^*(x) \neq \theta(x)\}) < \epsilon$, then $\varphi_\epsilon = \varphi^{-1} \circ \theta \circ \psi^{-1}$ is the required homeomorphism.

**Proof of Theorem 2.** Suppose $f$ is as hypothesized. For any Lebesgue measurable subset $A$ of $I^n$, let $\mu(A) = m(f^{-1}[A])$. Then $\mu \in M(I^n)$ and $f$ carries $m$ into $\mu$. Let $\delta > 0$ be such that $\delta \leq \epsilon$ and, if $m(A) < \delta$, then $\mu(A) < \epsilon$. By Theorem 1, there is a homeomorphism $\varphi_\epsilon$ of $I^n$ onto $I^n$ carrying $m$ into $\mu$ such that $m(\{x : f(x) \neq \varphi_\epsilon(x)\}) < \delta$. It is clear that $\varphi_\epsilon$ is the required homeomorphism.
Remarks. In proving Theorem 1 with \( \mu_1 = \mu_2 \), J. C. Oxtoby showed that \( \varphi_x \) could be chosen to be a homeomorphism of \( \text{cl } I^n \) onto \( \text{cl } I^n \) such that \( \varphi_x \) is equal to the identity outside of some closed interval contained in \( I^n \). It is clear that (a) the proof of Theorem 1 given here yields this, too, and (b) the \( \varphi_x \) in Theorem 2 may be chosen to have these properties. Furthermore, in Theorem 1, \( \varphi_x \) can be chosen to be a homeomorphism of \( \text{cl } I^n \) onto \( \text{cl } I^n \) which is equal to the identity on \( \text{bdry } I^n \).

References


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