A NOTE ON SPACES WITH NORMAL PRODUCT WITH SOME COMPACT SPACE

R. W. THOMASON

Abstract. For compact \( X \) with \( \log |X| \geq \aleph \), \( X \times Z \) is normal only if \( Z \) is \( \aleph \)-collectionwise normal. If \( Z \) is also semimetric or \( \aleph \)-metacompact, it is then \( \aleph \)-paracompact.

All spaces herein are to be Hausdorff. The density function is denoted \( d(\ ) \).

Lemma 1. If \( Y \) is a nondegenerate AE for \( X \), and \( Y^X \) has the compact-open topology, \( d(Y^X) \geq d(Y) \cdot \log |X| \).

Proof. \( \prod Y_x, x \in X \), has density \( d(Y) \cdot \log |X| \) [3, Theorem 4.5]. \( Y^X \) with the \( p \)-topology is a dense subspace of \( \prod Y_x \), so its density is no less. The result follows as the compact-open topology is finer than the \( p \)-topology.

Henceforth, \( Y^X \) will always have the compact-open topology. From Lemma 1, we see that if \( Y \) is a metric nondegenerate AE for normal spaces, and \( X \) is a compact space with \( |X| > 2^{\aleph_0} \), \( Y^X \) is not even an ANE for perfectly normal spaces. For \( d(Y^X) > \aleph_0 \), but \( Y^X \) is metric, and E. Michael has shown that metric ANE’s for Bing’s perfectly normal, but not uncountably collectionwise normal, spaces must be separable [6, Theorem 3.1]. (Incidentally, this shows the “only if” parts of Problems 5.7 and 5.8 of Chapter XV of Dugundji’s text [2] are incorrect.) We generalize this as follows:

Lemma 2. If \( Y \) is a metric ANE for \( Z \), \( Z \) is \( d(Y) \)-collectionwise normal.

Proof (cf. Proposition 5.1 of [6]). Since \( Y \) is metric, its cellularity is attained and equals its density [1, Theorem 4]. Given any discrete collection \( \{K_x\} \) of no more than \( d(Y) \) closed subsets of \( Z \), we may then take as many disjoint nonempty open subsets \( \{U_x\} \) of \( Y \). Let \( f \) be the continuous extension over an open neighborhood of \( \bigcup K_x \) of the map from...
∪ Kα to Y sending each Kα to a point in the corresponding Uα. Then \{f^{-1}(U_α)\} is a collection of disjoint open sets in Z separating \{K_α\}.

**Theorem.** Let X be compact, log |X| ≥ \aleph. Then if Z \times X is normal, Z is Ζ-collectionwise normal. If Z is also Ζ-metacompact or semimetric, it is then Ζ-paracompact.

**Proof.** Let A be a closed subset of Z. If f: A → RX is continuous, so is its associate f*: A \times X → R. We may extend f* to a continuous map F*: Z \times X → R, and its associate F: Z → RX is a continuous extension of f. Thus RX is an AE for Z; by Lemma 1, d(RX) ≥ Ζ; and now by Lemma 2, Z is Ζ-collectionwise normal.

Bearing in mind that point-finite refinements may be made precise [2, Theorem VIII 1.4], one may adapt Michael’s proof of Theorem 2 in [5] to show spaces both Ζ-metacompact and Ζ-collectionwise normal are Ζ-paracompact. Similarly, adapt McAuley’s proof of Lemma 2 in [4] to handle the case where Z is semimetric.

Note if log |X| ≥ 2d(Z) in the above, we get Z is collectionwise normal, for Z has no discrete collection of more than 2d(Z) subsets [2, VII 3, Example 3].

Morita has shown Z is Ζ-paracompact and normal iff Z \times I^Z is normal [7, Theorem 2.4]. The above theorem generalizes the “iff” part of this, replacing I^Z with any compact space of equal cardinality, but draws a weaker conclusion.

**References**


**Department of Mathematics, Michigan State University, East Lansing, Michigan 48823**

*Current address:* Department of Mathematics, Princeton University, Princeton, New Jersey 08540