AN EXAMPLE IN FIXED POINT THEORY
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Abstract. An example is given of a homeomorphism of $\mathbb{R}^2$ onto $\mathbb{R}^2$ which has no fixed points, such that each of its iterates has a fixed point.

A few years ago A. Granas posed the question: Does there exist a compact metric space $X$ and a mapping (i.e. continuous function) $f: X \to X$ with no fixed points, such that each iterate $f^n$ ($n \geq 2$) of $f$ has a fixed point?

W. Kuperberg has given as a simple solution (unpublished) the following: $X$ is the set of all points on the Cartesian plane whose polar coordinates $(p, \theta)$ satisfy the inequality $1 \leq p \leq 2$ and $f: X \to X$ by

$$f(p, \theta) = (p, \theta + (2 + p)\pi/3).$$

It is easy to see that $f$ is a homeomorphism of $X$ onto $X$ and that $f$ has no fixed points. Observe also that if $n$ is an integer greater than 1, then any point on the circle $p = p_n$ is a fixed point for $f^n$ where $p_n = 6K(n)/n - 2$ and $K(n)$ is an integer such that $\frac{1}{2} \leq K(n)/n \leq \frac{3}{2}$.

Our example gives an affirmative answer to the question: Is there a homeomorphism $h$ of the Cartesian plane $\pi$ onto itself which has no fixed point such that each iterate $h^n$ ($n \geq 2$) of $h$ has a fixed point?

To this end, define the homeomorphism $g$ from $\pi$ onto $\pi$ by $g(x, y) = (-x, y)$, let $W = \{(x, y) : |x| \leq 8\}$, and define $f$ on $\pi - W$ to be the identity function.

We now show how to define $f$ on $W$ such that $h = f \circ g$ is a homeomorphism of $\pi$ onto $\pi$ with the desired properties.

As an aid to this end, consider Figures 1 through 5.

In Figure 2 we indicate the image of a typical interval $[a, h]$ under $f$.

In Figure 3 we note that we require $c \rightarrow c'$, in Figure 4 that $b \rightarrow b'$, and in Figure 5 that $c$ maps onto $c'$. These are the essential actions of $f$ on $W$.

In Figure 1 we indicate the action of $f$ on a portion of $W$ and note that $h^3(P_3) = P_3$, $h^4(P_4) = P_4$ and $h^5(P_5) = P_5$.

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Figure 1
Figure 2

Figure 3

Figure 4
Figure 5

It is clear that $f$ can be defined on $W$ such that $h = f \circ g$ has the desired properties.

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