PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 45, Number 1, July 1974

A SHORT PROOF AND GENERALIZATION OF A MEASURE THEORETIC DISJOINTIZATION LEMMA

JOSEPH KUPKA

ABSTRACT. This paper presents general conditions under which a subfamily may be selected from an infinite family of nonnegative, finitely additive measures such that this subfamily has the same cardinality as the original family, and such that the members of this subfamily are, in a certain sense, disjointly supported. The generalized continuum hypothesis is required for the general result, but not for a special case of this result which had previously been obtained by Rosenthal, and for which the present techniques yield a much shorter proof.

This paper contains a short proof of a lemma about finitely additive measures which was first obtained by Rosenthal [2, Lemma 1.1, p. 16] in connection with a number of results on Banach spaces. Some illustration is then provided of the type of generalization which may be obtained with the present techniques.

We shall understand an *ordinal number* to be a set and not an order type (see [1, §4.3, p. 19]). As we shall rely heavily upon the axiom of choice, it will be sufficient for our purposes to define a *cardinal number* to be an *initial* ordinal number, i.e. an ordinal number which cannot be put into a one-to-one correspondence with any of its members. If Γ is any set, let $|\Gamma|$ denote the *cardinality* of Γ, by which we mean the unique cardinal number which can be put into a one-to-one correspondence with Γ; let $cf(\Gamma)$ denote the *cofinality* of $|\Gamma|$, by which we mean the smallest cardinal number κ such that $|\Gamma|$ contains a cofinal subset of cardinality κ; and let $P(\Gamma)$ denote the *power set* of Γ, by which we mean the set of all subsets of Γ.

To simplify notation we have trivially reworded Rosenthal's lemma, which now follows.

1. **Lemma.** *Let Γ be an infinite set, let $\{\mu_x: x \in \Gamma\}$ be a family of nonnegative finitely additive measures defined on all subsets of Γ, and assume*

Received by the editors June 25, 1973.

AMS (MOS) subject classifications (1970). Primary 28A10, 28–01, 04A20, 04–01; Secondary 46B99.

Key words and phrases. Finitely additive measures, cardinality.

that $\sup\{\mu_x(\Gamma): x \in \Gamma\}$ *is finite. Then, for all* $\epsilon > 0$, *there exists a set* $X \subset \Gamma$ *such that* $|X| = |\Gamma|$, *and such that* $\mu_x(X \sim \{x\}) < \epsilon$ *for all* $x \in X$.

Proof. Assume that, for some $\epsilon > 0$, no such set X exists. Since Γ is infinite, it is a well-known consequence of the axiom of choice that $|\Gamma| = |\Gamma \times \Gamma|$ [3, (2.2), p. 417], and hence that we have $\Gamma = \bigcup\{X_\gamma: \gamma \in \Gamma\}$, where the X_γ are pairwise disjoint, and where $|X_\gamma| = |\Gamma|$ for all $\gamma \in \Gamma$. We may now select an index $\gamma_0 \in \Gamma$ such that $\mu_x(\Gamma \sim X_{\gamma_0}) \geq \epsilon$ for all $x \in X_{\gamma_0}$. (For otherwise we could select, for all $\gamma \in \Gamma$, an $x_\gamma \in X_\gamma$ such that $\mu_{x_\gamma}(\Gamma \sim X_\gamma) < \epsilon$. The set $X = \{x_\gamma: \gamma \in \Gamma\}$ would then satisfy the conclusions of the Lemma, contrary to assumption.)

If this procedure is repeated with X_{γ_0} in place of Γ, and if it is iterated for at most finitely many steps, then the uniform boundedness of the μ_x is seen to be violated.□

Our generalization of this lemma is based upon a transfinite iteration of the procedure of the first paragraph of the above proof. In place of μ_x we shall consider N_x, the collection of sets of μ_x-measure zero, and in fact it will suffice to assume only that N_x is *hereditary*, i.e. that every subset of a member of N_x is also a member of N_x. Define $\delta(x)$ to be the smallest cardinal number δ such that every pairwise disjoint family of "x-nonnull" sets (i.e. sets which are $\notin N_x$) has cardinality $\leq \delta$. (Thus, in Lemma 1, we have $\delta(x) \leq \aleph_0$ for all $x \in \Gamma$.)

2. **Theorem.** *Assume the generalized continuum hypothesis. Let* Γ *be an infinite set, let* $\{N_x: x \in \Gamma\}$ *be a family of hereditary subsets of* $P(\Gamma)$, *and assume that there exists a cardinal number* $\kappa < cf(\Gamma)$ *such that* $\delta(x) < \kappa$ *for all* $x \in \Gamma$. *Then there exists a set* $X \subset \Gamma$ *such that* $|X| = |\Gamma|$, *and such that* $X \sim \{x\} \in N_x$ *for all* $x \in X$.

Remark. In particular, whenever $cf(\Gamma) \geq \aleph_2$, it is consistent with the axioms of set theory to replace "$< \epsilon$" by "$= 0$" in Lemma 1.

Proof. Assume that no such set X exists. Since Γ is infinite, it is a well-known consequence of the generalized continuum hypothesis that $|\Gamma| = |G|$, where G denotes the set of functions with domain κ and codomain Γ [1, §36.1, p. 162]. To simplify notation we shall treat the N_x as subsets of $P(G)$.

We now define a function $x^* \in G$ such that $\delta(x^*) \geq \kappa$, contrary to hypothesis. Let $\alpha \in \kappa$ be fixed, and assume by transfinite induction that $x^*(\beta)$ has been defined for all $\beta < \alpha$. Let $G_\alpha = \{x \in G: x(\beta) = x^*(\beta)$ for all $\beta < \alpha\}$, and observe that $G_\alpha = \bigcup\{X_\gamma: \gamma \in \Gamma\}$, where $X_\gamma = \{x \in G_\alpha: x(\alpha) = \gamma\}$

for all $\gamma \in \Gamma$. The argument of Lemma 1 is applicable, and we define $x^*(\alpha) = \gamma_0$, where γ_0 is any member of Γ such that $G_\alpha \sim X_{\gamma_0} \notin N_x$ for all $x \in X_{\gamma_0}$. Clearly, then, we have $\delta(x^*) \geq \kappa$, as desired.□

3. **Example.** Assume that $\Gamma = cf(\Gamma)$, and, for all $x \in \Gamma$, let N_x be the collection of those subsets of Γ which are disjoint from the predecessors of x. Then we have $\delta(x) = |x| < |\Gamma| = cf(\Gamma)$ for all $x \in \Gamma$, but it is easily shown that any set $X \subset \Gamma$ which satisfies the conclusions of Theorem 2 can contain at most a single point.

In particular, if $\Gamma = \aleph_0$ or $\aleph_1$, a trivial elaboration of this example will illustrate that, in general, the "$< \epsilon$" of Lemma 1 cannot be changed to "$= 0$" when $cf(\Gamma) = \aleph_0$ or $\aleph_1$.

To round out the picture we present a sample set of circumstances under which the hypotheses of Theorem 2 may be (slightly) relaxed.

4. **Proposition.** *Assume that each of the N_x of Theorem 2 is closed under the formation of finite unions, and that it contains every subset of Γ which has cardinality $< |\Gamma|$. Then, provided that $\delta(x) < cf(\Gamma)$ for all $x \in \Gamma$, there exists a set $X \subset \Gamma$ such that $|X| = |\Gamma|$, and such that $X \in N_x$ for all $x \in \Gamma$.*

Proof. By [3, Theorem 1, p. 451] (cf. [2, Proposition, p. 23]), there exists a set $S \subset P(\Gamma)$ such that $|S| > |\Gamma|$, such that $|E| = |\Gamma|$ for all $E \in S$, and such that $|E \cap F| < |\Gamma|$ whenever E and F are distinct members of S. It is easily established that, for all $x \in \Gamma$, there must be (strictly) fewer than $cf(\Gamma)$ ($\leq |\Gamma|$) many sets $E \in S$ such that $E \notin N_x$. It follows that there must exist at least one (and in fact $> |\Gamma|$ many) $X \in S$ such that $X \in N_x$ for all $x \in \Gamma$. □

5. **Example.** Assume that $|\Gamma| = cf(\Gamma)$, and, for all $x \in \Gamma$, let N_x be the collection of those subsets of Γ which have cardinality $< |\Gamma|$. Then we have $\delta(x) = |\Gamma| = cf(\Gamma)$ for all $x \in \Gamma$, whereas no set $X \in N_x$ has $|X| = |\Gamma|$.

REFERENCES

1. H. Bachmann, *Transfinite Zahlen*, Ergebnisse der Math. und ihrer Grenzgebiete, Heft 1, Springer-Verlag, Berlin, 1955. MR 17,134.

2. H. P. Rosenthal, *On relatively disjoint families of measures, with some applications to Banach space theory*, Studia Math. 37 (1970), 13–36. MR 42 #5015.

3. W. Sierpiński, *Cardinal and ordinal numbers*, 2nd rev. ed., Monografie Mat., vol. 34, PWN, Warsaw, 1965. MR 33 #2549.

DEPARTMENT OF MATHEMATICS, MONASH UNIVERSITY, CLAYTON, VICTORIA, AUSTRALIA