ON A SUBCLASS OF BAZILEVIĆ FUNCTIONS

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ABSTRACT. The authors show that certain Bazilević functions are spiral-like. Then the authors study the growth and Hardy classes of those special functions.

Introduction. I. E. Bazilević [2] gave an explicit construction for a class of functions analytic and univalent in the unit disc $D$ (see also [10]). His result was as follows.

Theorem 1. Let $g$ be univalent starlike in $D$ with $g(0) = 0$, and let $h$ be analytic and satisfy $\text{Re}(e^{i\lambda}h(z)) > 0$ in $D$ for some real $\lambda$. Then if $a > 0$ and $\beta$ is real, the function

$$f(z) = \left\{ \int_0^z g^a(\zeta)h(\zeta)\zeta^i\beta-1 d\zeta \right\}^{1/(a+i\beta)}$$

is analytic and univalent in $D$.

In this paper we consider the functions $f$ that arise from (1) when $h(z) \equiv 1$. Such a function $f$ must satisfy

$$\text{Re}\left[ 1 + z/f''(z)/f'(z) + (a - 1 + i\beta)zf'(z)/f(z) \right] > 0, \quad z \in D.$$ 

Conversely, if $f$ is analytic in $D$, with $f(0) = 0$, $f(z)f'(z)/z \not\equiv 0$ ($z \in D$), and if $f$ satisfies (2) for some $a > 0$, $\beta$ real, then $f$ can be written in the form (1), with $h(z) \equiv 1$. Let us denote the class of such functions $f$ by $B(a + i\beta)$. The class obtained when $\beta = 0$ has been studied extensively [5], [6], [7], [8], [9]. The class $B(1 + i\beta)$ has recently been considered by H. Yoshikawa [12]; he showed that if $f \in B(1 + i\beta)$ then $f$ is $\gamma$-spiral-like, where $\gamma = \text{arc tan} \beta$. We generalize this to

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Theorem 2. If $f \in B(\alpha + i\beta)$ then $f$ is $\gamma$-spiral-like, where $\gamma$ satisfies
\[ \alpha + i\beta = (\alpha^2 + \beta^2)^{1/2} e^{i\gamma}, \quad -\pi/2 < \gamma < \pi/2. \]

Proof. Define a function $w$ analytic in $D$ by
\[ e^{i\gamma} \frac{zf'(z)}{f(z)} = \cos \gamma \left( \frac{1 + w(z)}{1 - w(z)} \right) + i \sin \gamma, \quad z \in D. \]

One easily checks that $w(0) = 0$, $w(z) \neq \pm 1$ ($z \in D$). It suffices to show $|w(z)| < 1$ for $z \in D$. Let $w(z) = R(z)e^{i\phi(z)}$ for $z = re^{i\theta}$ and suppose that $z_0 = r_0e^{i\theta_0}$ is a point of $D$ such that
\[ \max_{|z| \leq r_0} |w(z)| = |w(z_0)| = 1. \]

Then $(\partial R/\partial \theta)|_{z=z_0} = 0$, and since
\[ \frac{zw'(z)}{w(z)} = \frac{\partial \phi}{\partial \theta} - i \frac{1}{R} \frac{\partial R}{\partial \theta} \]
we have at the point $z_0$,
\[ z_0w'(z_0)/w(z_0) = (\partial \phi/\partial \theta)|_{z=z_0}. \]

We shall now show that
\[ \Re \left[ 1 + zf''(z)/f'(z) + (\alpha - 1 + i\beta)zf'(z)/f(z) \right]_{z=z_0} = 0 \]
thus contradicting the assertion $f \in B(\alpha + i\beta)$. By (3), (4) can be written as
\[ \Re \left[ \frac{zP'(z)}{P(z) + i \tan \gamma} + (\alpha^2 + \beta^2)^{1/2} (\cos \gamma P(z) + i \sin \gamma) \right]_{z=z_0} = 0 \]
where $P(z) = (1 + w(z))/(1 - w(z))$. Since $|w(z_0)| = 1$ and since $[z_0w'(z_0)/w(z_0)]$ is real, it follows that $P(z_0)$ is imaginary and $z_0P'(z_0)$ is real. Hence (5) holds at $z_0$. This completes the proof.

Theorem 3. If $\alpha' + i\beta' = t(\alpha + i\beta)$, $t \geq 1$, then $B(\alpha + i\beta) \subset B(\alpha' + i\beta')$.

Proof. Since $f$ is $\gamma$-spiral-like ($\alpha + i\beta = (\alpha^2 + \beta^2)^{1/2} e^{i\gamma}$),
\[ \Re \left[ (t - 1)(\alpha^2 + \beta^2)^{1/2} e^{i\gamma} zf'(z)/f(z) \right] \geq 0, \quad z \in D. \]

Then
\[ \Re \left[ 1 + zf''(z)/f'(z) + (\alpha' - 1 + i\beta')zf'(z)/f(z) \right] = \Re \left[ 1 + zf''(z)/f'(z) + (\alpha - 1 + i\beta)zf'(z)/f(z) \right] \]
\[ + \Re \left[ (t - 1)(\alpha^2 + \beta^2)^{1/2} e^{i\gamma} zf'(z)/f(z) \right] \geq 0, \quad z \in D. \]
In the integral representation for functions in $B(\alpha + i\beta)$, namely,

$$f(z) = \left\{ \int_0^z g(\zeta)\zeta^{i\beta - 1} \, d\zeta \right\}^{1/(\alpha + i\beta)} ,$$

let us denote by $f_{\alpha+i\beta}$ the function obtained by letting $g$ be the Koebe function $z/(1-z)^2$. The following theorem illustrates the dependence of the growth of $f$ on the parameters $\alpha$ and $\beta$.

**Theorem 4.** Suppose $f \in B(\alpha + i\beta)$.

(A) If $0 < \alpha \leq \frac{1}{2}$, then, unless $f$ is a rotation or magnification of $f_{1/2+i\beta}$, $f$ is bounded.

(B) If $\alpha > \frac{1}{2}$ and $f$ is not a rotation or magnification of $f_{\alpha+i\beta}$, then there exists $\epsilon = \epsilon(f) > 0$ such that $f \in H^{\lambda+\epsilon}$ and $f' \in H^{(1+\lambda)\epsilon}$, where $\lambda = (\alpha^2 + \beta^2)/(2\alpha - 1)$.

(C) For $\alpha > \frac{1}{2}$ the function $f_{\alpha+i\beta}$ belongs to $H^p$, $\forall p < \lambda$, but does not belong to $H^\lambda$.

**Proof.** Following Sheil-Small’s construction [11] of $f$ in “analytic stages” from the representation (6), we set

$$F(z) = \left( \frac{g(z)}{z} \right)^{\alpha} = \sum_{n=0}^{\infty} C_n z^n,$$

for a suitable branch of the nonvanishing function $(g(z)/z)^\alpha$. Let

$$G(z) = \sum_{n=0}^{\infty} \frac{C_n}{n + \alpha + i\beta} z^n,$$

so that $G$ is analytic in $D$ and satisfies the differential equation

$$(\alpha + i\beta)G(z) + zG'(z) = F(z).$$

Sheil-Small [11] shows that $G(z) \neq 0$ in $D$. We now define $f$ by the formula

$$f(z) = z[G(z)]^{1/(\alpha + i\beta)}.$$

One can easily verify that apart from a constant factor, this defines an analytic branch of the formula (6). By (9) we may write

$$G(z) = [(f(z)/z)^{1+i\beta/\alpha}]^\alpha = [s(z)/z]^{\alpha},$$

where (10) is the defining equation for $s$. Since $f$ is $\gamma$-spiral-like (Theorem 2), it follows easily that $s$ is starlike in $D$. From (8) we have

$$zG'(z) = (g(z)/z)^\alpha - (\alpha + i\beta)(s(z)/z)^\alpha.$$

If $g$ is not a rotation of the Koebe function, then there exists $\epsilon = \epsilon(g) > 0$ such that $g \in H^{1/2+\epsilon}$ [4]. Furthermore, it is easy to see from (7) that $s$ cannot be a rotation of the Koebe function. Thus, $G' \in H^{1/(2\alpha)+\epsilon}$, $\epsilon$ denoting
a positive number, not necessarily the same in each instance. Hence, if
0 < α ≤ 1/2, G is bounded and so f is bounded, by (9). Whence, (A) is proved.
For α > 1/2, a Hardy-Littlewood theorem [3, p. 88] shows that G ∈ H^{1/(2α−1)+ε}; hence, from (10), s ∈ H^{a/(2α−1)+ε}. From the identity
\[ (f(z)/z)^{1+iβ/α} = s(z)/z, \]
we have
\[ \left| \frac{f(z)}{z} \right| = \left| \frac{s(z)}{z} \right|^2 \left( \alpha^2 + \beta^2 \right)^{-1/2} \exp \left[ \frac{\alpha \beta}{\alpha^2 + \beta^2} \arg \left( \frac{s(z)}{z} \right) \right]. \]
Since s ∈ H^{a/(2α−1)+ε} and the exponential factor is bounded, it follows that
\[ f ∈ H^{λ+ε}, \quad λ = \frac{\alpha^2 + \beta^2}{\alpha(2α - 1)}. \]
To complete the proof of (B), we must show the existence of an ε > 0 such that f' ∈ H^{(1+λ)^{-1}+ε}. By Theorem 2, there exists h, Re(h(z)) > 0 in D, such that
\[ e^{iγ} z f'(z) = f(z) h(z). \]
Fix ε in (14), and for small positive δ, let
\[ k = k(δ) = (λ + ε - δλ + δε)^{-1}. \]
Choosing p = (λ + ε)k^{-1}, q = (1 + δ)^{-1}k^{-1}, and applying Hölder’s inequality to (15) with conjugate indices p and q, it follows that
\[ \int_{-π}^{π} |f'(z)|^k dθ ≤ \left( \int_{-π}^{π} \left| \frac{f(z)}{z} \right|^p dθ \right)^{1/p} \left( \int_{-π}^{π} |h(z)|^q dθ \right)^{1/q}. \]
By (14) and the fact that kq < 1, we have that \( \int_{-π}^{π} |f'(z)|^k dθ \) remains bounded as r→1. Hence, the proof of (B) is complete if we show the existence of δ > 0 such that k = k(δ) > λ(1 + λ)^{-1}. But this is easily checked by consideration of (16).
Finally, we leave the verification of (C) to the reader.

Remark 1. If we take β = 0 in Theorem 4, the result is the same as that obtained in [5].

Remark 2. Note that if we divide (2) by \( (\alpha^2 + \beta^2)^{1/2} \) and let \( α + iβ → ∞ \) along the ray \( te^{iγ} \) the classes \( B(α + iβ) \) "tend" to the full class of γ-spiral-like functions. If we also let \( α + iβ → ∞ \) in Theorem 4, we find that \( λ = λ(α, β) → (2 \cos^2 γ)^{-1} \); and this is precisely the Hardy class result.
previously known for $\gamma$-spiral-like functions [1].

Remark 3. It is interesting to observe that the level curves of $\lambda = (\alpha^2 + \beta^2)[\alpha(2\alpha - 1)]^{-1}$ are the right branches of certain hyperbolas that are symmetric with respect to the real line; and they converge on the vertical line $\alpha = \frac{1}{2}$ (as $\lambda \to \infty$). For instance, if $\lambda = 1$ we obtain the right branch of the hyperbola $4(\alpha - \frac{1}{2})^2 - 4\beta^2 = 1$.

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