THE RADON-NIKODYM PROPERTY AND DENTABLE SETS IN BANACH SPACES

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ABSTRACT. In order to prove a Radon-Nikodym theorem for the Bochner integral, Rieffel [5] introduced the class of "dentable" subsets of Banach spaces. Maynard [3] later introduced the strictly larger class of "s-dentable" sets, and extended Rieffel's result to show that a Banach space has the Radon-Nikodym property if and only if every bounded nonempty subset of E is s-dentable. He left open, however, the question as to whether, in a space with the Radon-Nikodym property, every bounded nonempty set is dentable. In the present note we give an elementary construction which shows this question has an affirmative answer.

Definitions. A Banach space E has the Radon-Nikodym property if for each totally finite positive measure space (X, Σ, μ) and each E-valued, μ-continuous measure  on X to E such that μ(E) = ∫E f dμ (E ∈ Σ).

A subset A of E is dentable if for every ε > 0, there exists x ∈ A such that x ∈ cl Λ(A \ S_ε(x)). [Here co B denotes the convex hull of B, clcoB is its closure and S_ε(x) = {y ∈ E: ∥x − y∥ ≤ ε}.] A bounded set A is called s-dentable if for each ε > 0 there exists x ∈ A such that x ∈ s(A \ S_ε(x)). [Here s(B) = {∑_i=1^∞ λ_i x_i: ∑ λ_i = 1, {x_i} ⊂ B}.] A point x ∈ A is a denting [s-denting] point if for all ε > 0, x ∈ cl Λ(A \ S_ε(x)) [x ∈ s(A \ S_ε(x))].

Dentable sets are s-dentable, and Maynard has given an example of a bounded set which is s-dentable but not dentable. Rieffel has shown that if A is not dentable, then neither is clco A. The analogous assertion fails for s-dentability: The closed unit ball of C([0, 1]) is not dentable [5], but the constant-ly 1 function is an s-denting point. By Lemma 2 below, its interior is not s-dentable.

Lemma 1. A subset A of E is not dentable if and only if there exists
$\epsilon > 0$ such that $A \subseteq \text{cl}\, \text{co}(A \setminus S_\epsilon(x))$ for each $x \in A$. If $A$ is closed and convex, this is equivalent to $A = \text{cl}\, \text{co}(A \setminus S_\epsilon(x))$ for each $x \in A$.

**Proof.** One implication is trivial. For the other, suppose $A$ is not dentable. Then there exists $\gamma > 0$ such that for each $y \in A$, $y \in \text{cl}\, \text{co}(A \setminus S_{\gamma}(y))$. Suppose that $x, y \in A$ and $\|x - y\| > \epsilon$. Then $y \in \text{cl}\, \text{co}(A \setminus S_{\gamma}(x))$. On the other hand, if $\|x - y\| \leq \epsilon$, then $S_\epsilon(x) \subseteq S_{\gamma}(y)$ so that $y \in \text{cl}\, \text{co}(A \setminus S_{2\epsilon}(y)) \subseteq \text{cl}\, \text{co}(A \setminus S_\epsilon(x))$, completing the proof.

**Lemma 2.** Suppose $C$ is a closed convex set in $E$ with nonempty interior (denoted by $\text{int}\, C$) and suppose $C$ is not dentable. Then there exists $\epsilon > 0$ such that for each $x \in C$, $\text{int}\, C \subseteq \text{co}\, [\text{int}\, C \setminus S_\epsilon(x)]$. In particular, $\text{int}\, C$ is not s-dentable.

**Proof.** By Lemma 1, there exists $\epsilon > 0$ such that $C = \text{cl}\, \text{co}(C \setminus S_\epsilon(x))$ for each $x \in C$. Let $J_x = C \setminus S_\epsilon(x)$ so that $\text{int}\, J_x = \text{int}\, C \setminus S_\epsilon(x)$. For $\epsilon$ sufficiently small, $\text{int}\, J_x \neq \emptyset$ for each $x \in C$. Fix $x$ and let $J = J_x$. Then $C = \text{cl}\, \text{co}\, J$ and we want to show that $\text{int}\, (\text{cl}\, \text{co}\, J) \subseteq \text{co}\, (\text{int}\, J)$. Note that $J \subseteq \text{cl}\, (\text{int}\, J)$: If $y \in J$, then $y \in C$ and $\|y - x\| > \epsilon$. Let $z \in \text{int}\, C$, so that $[z, x) \subseteq \text{int}\, C$ and there exists $u \in [z, x) \cap S_\epsilon(x)$. Therefore, $[u, y) \subseteq \text{int}\, C$ so for some $v \in [u, y)$ we have $[v, y) \subseteq \text{int}\, C \setminus S_\epsilon(x)$. Thus $y \in \text{cl}\, (\text{int}\, J)$. It now follows that $\text{co}\, J \subseteq \text{cl}\, \text{co}(\text{int}\, J)$. Taking the interior of each side, we conclude that

$$\text{int}\, (\text{cl}\, \text{co}\, J) = \text{int}\, (\text{co}\, J) \subseteq \text{int}\, (\text{cl}\, \text{co}(\text{int}\, J)) = \text{co}\, (\text{int}\, J).$$

The equalities follow from the fact that the interior of a convex set coincides with the interior of its closure.

**Proposition.** If $E$ contains a bounded nonempty set which is not dentable, then it contains a bounded closed convex and symmetric set $C$ which is not dentable and which has nonempty interior. In particular, $E$ can be renormed so that the new unit ball is not dentable and the interior of the new unit ball is not s-dentable.

**Proof.** If $A$ is a bounded nonempty set which is not dentable, then the same is true of the sets $A_1 = A \cup (-A)$ (definition), $A_2 = \text{cl}\, \text{co}\, A_1$ (Rieffel [5, Proposition 2]) and $A_3 = S + A_2$, where $S$ is the closed unit ball of $E$ (easy computation). Let $C$ be the closure of $A_3$. Again, by Rieffel's proposition, $C$ is not dentable. By Lemma 2, $\text{int}\, C$ is not s-dentable.

What we have shown is that every bounded subset of $E$ is dentable if and only if every bounded subset of $E$ is s-dentable. This yields the following corollary.
Corollary: A Banach space $E$ has the Radon-Nikodym property if and only if every bounded nonempty set in $E$ is dentable.

J. Diestel has raised the question of the relationship between the Radon-Nikodym property and the Krein-Milman property [every closed bounded convex set is the closed convex hull of its extreme points]. It is known that a space has both properties if it is reflexive or is a separable conjugate space [1], [4]. The Proposition shows that every bounded nonempty subset of $E$ is dentable if and only if the unit ball of every Banach space isomorphic to $E$ is dentable. Is the analogous assertion true for the Krein-Milman property? That is, if $E$ contains a closed, bounded nonempty convex set which is not the closed convex hull of its extreme points, can $E$ be renormed so that its new unit ball is not the closed convex hull of its extreme points? The answer is affirmative for the spaces $(m)$ and $L_1([0, 1])$ [2]. In spaces with the Radon-Nikodym property does every closed bounded convex set have a denting point (s-denting point, extreme point)?

Added in proof. Since this paper was submitted, a number of related results have appeared. Huff [6] has given an independent proof of the above corollary, by modifying the proof of Maynard's main theorem [3]. Lindenstrauss has used the corollary to show (cf. [7]) that the Radon-Nikodym property implies the Krein-Milman property, and the question concerning denting points has also received an affirmative answer in [7]. Huff and Morris [8] have shown that the Krein-Milman property implies the Radon-Nikodym property in conjugate spaces, but this implication remains open for arbitrary spaces. Finally, Edgar [9] has generalized Lindenstrauss' result by obtaining a Choquet-type representation theorem for bounded closed convex subsets of a Banach space with the Radon-Nikodym property.

BIBLIOGRAPHY


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