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A NOTE ON THE DUGUNDJI EXTENSION THEOREM

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ABSTRACT. We prove that if A is a closed, metrizable, G_δ-subspace of a collectionwise normal space X then there is a linear transformation $e: C(A) \longrightarrow C(X)$ such that for each $g \in C(A)$, $e(g)$ extends g and the range of $e(g)$ is contained in the closed convex hull of the range of g.

For any space Y let $C(Y)$ denote the vector space of all continuous real-valued functions on Y. Let A be a subspace of a space X. By a *simultaneous extender from A to X* we mean a linear transformation $e: C(A) \longrightarrow C(X)$ which has the property that for each $g \in C(A)$, $e(g)$ extends g and the range of $e(g)$ is contained in the closed convex hull of the range of g. Using this terminology, the well-known Dugundji extension theorem [3] asserts that if X is metrizable and A is a closed subset of X then simultaneous extenders from A to X exist. In this note we prove:

1. **Theorem.** *If X is collectionwise normal and A is a closed, metrizable, G_δ-subspace of X then simultaneous extenders from A to X exist.*

Our proof uses the pseudo-metric extension theorem of R. Arens [2] as generalized by R. Alo [1]:

2. **Proposition.**[2] *Let A be a closed metrizible subspace of a collectionwise normal space (X, T). Let d be a metric on the set A which induces on A the relative topology which A inherits from X. Then there is a pseudo-metric D on X which extends the metric d and which generates a topology M on X having $M \subset T$.*

The second major ingredient in our proof is a slight generalization of the original Dugundji extension theorem wherein we consider pseudo-metriz-

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2 Proposition (2) is also implicit in Dowker's paper *On a theorem of Hanner* [Ark. Mat. **2** (1952), 307–313] as was pointed out by E. Michael in Math. Rev. 14 (1953), 396.

able spaces [5] instead of metrizable spaces.

3. **Proposition.** *Suppose A is a closed subspace of a pseudo-metrizable space X. Then simultaneous extenders from A to X exist.*

Proof. Fix a pseudo-metric on X and let $\tilde{X}$ be the space obtained from X by identifying all points at zero-distance from one another. Then $\tilde{X}$ is metrizable and the set $\tilde{A}$, the image of A under the quotient map $\pi: X \to \tilde{X}$, is closed in $\tilde{X}$. Further, each function $g \in C(A)$ determines a unique function $\tilde{g} \in C(\tilde{A})$. Letting $\tilde{e}: C(\tilde{A}) \to C(\tilde{X})$ be a simultaneous extender, we define $e: C(A) \to C(X)$ by the rule that for each $g \in C(A)$ and $x \in X$, $e(g)(x) = \tilde{e}(\tilde{g})(\pi(x))$.

We may now prove Theorem 1 as follows. Supposing A is a closed G_δ set in the (collectionwise) normal space (X, T), A is the zero set of some continuous real-valued function $F: X \to \mathbf{R}$. Let d be a metric compatible with the topology T_A on A; according to Proposition 2, we may extend d to a pseudo-metric D on X whose induced pseudo-metric topology M satisfies $M \subset T$. We define

$$r(x, y) = D(x, y) + |F(x) - F(y)|,$$

thereby obtaining a continuous pseudo-metric on X with respect to which A is closed. Further, the topology M_A induced by r on A coincides with the topology T_A which A inherits from (X, T). According to Proposition 3 there is a simultaneous extender $e: C(A, M_A) \to C(X, M)$. Since $C(X, M) \subset C(X, T)$ and since $C(A, M_A) = C(A, T_A)$, e is seen to be the required simultaneous extender.

Remark. This note grew out of an attempt to verify an assertion appearing on p. 806 of [6] where it is asserted that simultaneous extenders from A to X will exist provided A is a metrizable closed subset of a paracompact space X (the hypothesis that A is also a G_δ was inadvertently omitted). An example due to Heath and Lutzer shows that this additional hypothesis is necessary: in [4] they show that if X is the Michael line [7] and A is the closed subspace of X consisting of all rational numbers, then no simultaneous extender from $C(A)$ to $C(X)$ can be found.

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