PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 45, Number 1, July 1974

ON FUNDAMENTAL TRANSVERSAL MATROIDS[1]

R. A. BRUALDI

ABSTRACT. Unified proofs of two theorems on fundamental transversal matroids are presented. A necessary condition for a matroid to be a fundamental transversal matroid with respect to a given basis is given.

The purpose of this note is to prove a theorem about fundamental transversal matroids which in turn yields unified proofs of two recent theorems about such matroids. We assume the reader has a modest acquaintance with the basic notions of matroid theory. Thus he or she should know that a *matroid* (or *combinatorial pregeometry*) $\mathbf{M}$ on a finite set E consists of a nonempty, hereditary collection of subsets of E which are called *independent sets*, and that the maximal independent sets all have the same cardinality (the *rank* of $\mathbf{M}$) and are called *bases*. Corresponding to a matroid $\mathbf{M}$ there are *circuits* (minimal nonindependent sets) and a *dual matroid* $\mathbf{M}^*$ whose bases (*cobases* of $\mathbf{M}$) are the complements of the bases of $\mathbf{M}$. The circuits of $\mathbf{M}^*$ are called *cocircuits* (or *bonds*) of $\mathbf{M}$. A property important in what follows is that a cocircuit and circuit cannot have exactly one element in common. A matroid is *nonseparable* if every pair of elements of E lie in a common circuit; otherwise there is a partition E_i $(1 \leq i \leq t)$ of E and matroids $\mathbf{M}_i$ on E_i $(1 \leq i \leq t)$ such that $\mathbf{M}$ is the *direct sum* $\mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_t$. For more details one can consult [2], [4], or [7].

An interesting kind of matroid is a *transversal matroid* [3] whose independent sets are the partial transversals of a family of sets. If $(A_i: i \in I)$ is a family of subsets of set E, then $\mathbf{T}(A_i: i \in I)$ denotes the corresponding transversal matroid.

Suppose $\mathbf{M}$ is a matroid on E and B is a basis of $\mathbf{M}$. Each $e \in B$ gives rise to a unique cocircuit C_e^* with $e \in C_e^* \subseteq (E \setminus B) \cup \{e\}$ called the *fundamental cocircuit* of e with respect to the cobasis $E \setminus B$. For $e \in E \setminus B$

Received by the editors October 12, 1972 and, in revised form, June 22, 1973.

AMS (MOS) subject classifications (1970). Primary 05B35; Secondary 05A05, 05C99.

[1] Research supported in part by National Science Foundation Grant No. GP-17815.

set $C_e^* = \{e\}$. Dually each $x \in E \setminus B$ gives rise to a unique circuit C_x with $x \in C_x \subseteq B \cup \{x\}$, called the *fundamental circuit* of x with respect to the basis B. For $x \in B$ set $C_x = \{x\}$. The matroid $\mathbf{M}$ is a *fundamental transversal matroid* [1] if there is a basis B such that $\mathbf{M} = \mathbf{T}(C_e^* : e \in B)$. Examples of such matroids are the so-called free matroids $\mathcal{P}_r(E)$ whose bases consist of all r element subsets of E (any basis works for B in the definition) and the cycle matroid [4] of the graph of Figure 1.

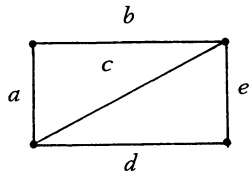

Figure 1

A basis (spanning tree) B can be used to show this is a fundamental transversal matroid if and only if $c \in B$. For example, if $B = \{a, c, e\}$, then $\{a, b\}$, $\{b, c, d\}$, and $\{d, e\}$ give a fundamental presentation. After proving some results we shall look at this example again.

Theorem 1. *Suppose* $\mathbf{M}$ *is a matroid on* E *with rank* r. *Let* B *be a basis of* $\mathbf{M}$, *and* F *any* r *element subset of* E. *Then* F *is an independent set (i.e. basis or transversal) of* $\mathbf{T}(C_e^*: e \in B)$, *denoted* $F \in \mathbf{T}(C_e^*: e \in B)$, *if and only if* $B \in \mathbf{T}(C_x : x \in F)$.

Proof. Suppose $B \in \mathbf{M}(C_x : x \in F)$, say $B = \{b_x : b_x \in C_x,\ x \in F\}$. If for some $b_x \in B \setminus F$, $x \notin C_{b_x}^*$, then $|C_{b_x}^* \cap C_x| = |\{b_x\}| = 1$. Since this cannot happen, $x \in C_{b_x}^*$ for all $x \in F \setminus B$. If $x \in F \cap B$, then $x \in C_x^*$, so $F \in \mathbf{T}(C_e^*: e \in B)$. The converse is proved in a dual manner.

Applying Theorem 1 to the dual matroid $\mathbf{M}^*$ of $\mathbf{M}$ we obtain

Theorem 1*. *If* B *is a basis of the matroid* $\mathbf{M}$ *of rank* r *on* E *and* F *is an* r *element subset of* E, *then* $E \setminus F \in \mathbf{T}(C_x : x \in E \setminus B)$ *if and only if* $E \setminus B \in \mathbf{T}(C_e^* : e \in E \setminus F)$.

For matroids $\mathbf{M}, \mathbf{M}'$ of equal rank on the set E, recall that $\mathbf{M} \subseteq \mathbf{M}'$ means that all independent sets (equivalently bases) of $\mathbf{M}$ are independent sets of $\mathbf{M}'$. In the terminology of [2] this means that the identity map on E is a geometric (weak) map.

Corollary 1. $\mathbf{M} \subseteq \mathbf{T}(C_e^*: e \in B)$.

Proof. Let F be any basis of $\mathbf{M}$. Suppose $B \notin \mathbf{T}(C_x : x \in F)$. Then by

Hall's theorem (see [6]), there exists $G \subseteq F$ with $|\bigcup_{x \in G} C_x \cap B| < |G|$. But then $\bigcup_{x \in G} C_x \cap B$ and G are both independent subsets of $\bigcup_{x \in G} C_x$ with the former a maximal independent subset and the latter of larger cardinality. This contradicts a basic tenet of matroid theory. The proof is now completed by invoking Theorem 1.

Corollary 2. *Let F be an r element subset of E which is not a basis of* $\mathbf{M}$. *Then there is a basis B of* $\mathbf{M}$ *(any extension to a basis of a maximal independent subset A of F will do) such that $F \notin \mathbf{T}(C_e^*: e \in B)$.*

Proof. $C_x \subseteq A$ for all $x \in F$. Hence $B \notin \mathbf{T}(C_x: x \in F)$, so by Theorem 1, $F \notin \mathbf{T}(C_e^*: e \in B)$.

The matroid $\mathbf{M}$ is the basis intersection of matroids $\mathbf{M}_1, \cdots, \mathbf{M}_t$ provided the bases of $\mathbf{M}$ are precisely the common bases of $\mathbf{M}_1, \cdots, \mathbf{M}_t$.

Corollary 3 (Bondy and Welsh [1]). *Every matroid of rank r is the basis intersection of fundamental transversal matroids of rank r.*

Proof. Corollaries 1 and 2.

Theorem 2. *Let F be an r element subset of E. Then $F \in \mathbf{T}(C_e^*: e \in B)$ if and only if $E \backslash B \in \mathbf{T}(C_e^*: e \in E \backslash F)$.*

Proof. Since for $x \in F \cap B$, the only member of the family $(C_e^*: e \in B)$ which contains x is C_x^*, we conclude that

(i) $F \in \mathbf{T}(C_e^*: e \in B)$ if and only if $F \backslash B \in \mathbf{T}(C_e^*: e \in B \backslash F)$. Now $E \backslash B = (F \backslash B) \cup X$ and $E \backslash F = (B \backslash F) \cup X$ where $X = (E \backslash F) \cap (E \backslash B)$. Since for $e \in X$, $C_e^* = \{e\}$ we conclude that

(ii) $E \backslash B \in \mathbf{M}(C_e^*: e \in E \backslash F)$ if and only if $F \backslash B \in \mathbf{T}(C_e^*: e \in B \backslash F)$. Combining (i) and (ii) we obtain the conclusion of the theorem.

Corollary 4. *Let F be an r element subset of E. Then $B \in \mathbf{T}(C_x: x \in F)$ if and only if $E \backslash B \in \mathbf{T}(C_e^*: e \in E \backslash F)$.*

Proof. Theorems 1 and 2.

Corollary 5 (Las Vergnas [5]). *The dual of a fundamental transversal matroid is a fundamental transversal matroid.*

Proof. Let $\mathbf{M} = \mathbf{T}(C_e^*: e \in B)$. Let F be any r element subset of E. Then

$$\begin{aligned} E\backslash F \in \mathbf{M}^* &\Leftrightarrow F \in \mathbf{M} \\ &\Leftrightarrow F \in \mathbf{T}(C_e^*\colon e \in B) \\ &\Leftrightarrow E\backslash B \in \mathbf{T}(C_e^*\colon e \in E\backslash F) \quad \text{(Theorem 2)} \\ &\Leftrightarrow E\backslash F \in \mathbf{T}(C_x\colon x \in E\backslash B) \quad \text{(Theorem 1*)}. \end{aligned}$$

Thus $\mathbf{M}^* = \mathbf{T}(C_x\colon x \in E\backslash B)$.

Example. Let $\mathbf{M}$ be the cycle matroid of the graph which is obtained from a triangle by doubling each edge. It is well known and easy to check that $\mathbf{M}$ is not a transversal matroid. In light of Corollary 5 this means that $\mathbf{M}^*$ is not a fundamental transversal matroid (although it is a transversal matroid), since $\mathbf{M} = (\mathbf{M}^*)^*$.

Theorem 3. *Let $\mathbf{M}$ be a matroid on E and let B be a basis. Suppose for some $y \in E\backslash B$ there is a circuit D with $y \in D$ and $|D| < |C_y|$. Then $\mathbf{M} \neq \mathbf{T}(C_e^*\colon e \in B)$.*

Proof. Consider the family $(C_x \cap B\colon x \in D)$. Since every proper subset A of D is independent, for all $A \subseteq D$, $|\bigcup_{x \in A} C_x \cap B| \geq |A|$. But also

$$\left|\bigcup_{x \in D} C_x \cap B\right| \geq |C_y \cap B| = |C_y| - 1 \geq |D|.$$

Thus by Hall's theorem there is a subset T of B which is a transversal of $(C_x \cap B\colon x \in D)$. Let F be the r element set $D \cup (B\backslash T)$. Then since $C_x = \{x\}$ for $x \in B\backslash T$, $B \in \mathbf{T}(C_x\colon x \in F)$. So by Theorem 1, $F \in \mathbf{T}(C_e^*\colon e \in B)$. Since $D \subseteq F$ and D is a circuit of $\mathbf{M}$, $\mathbf{M} \neq \mathbf{T}(C_e^*\colon e \in B)$.

If $\mathbf{M}$ is the cycle matroid of the graph of Figure 1, then while $\mathbf{M}$ is a fundamental transversal matroid $\mathbf{M} \neq \mathbf{T}(C_x^*\colon x \in B)$ for the basis $B = \{a, b, e\}$. We can say this since $C_d = \{a, b, d, e\}$ while $D = \{d, c, e\}$ is a smaller circuit containing d.

As a final result we characterize those matroids which are fundamental transversal matroids with respect to every basis.

Theorem 4. *Let $\mathbf{M}$ be a nonseparable matroid of rank r on E such that $\mathbf{M} = \mathbf{T}(C_e^*\colon e \in B)$ for all bases B. Then $\mathbf{M}$ is the matroid $\mathcal{P}_r(E)$.*

Proof. From Theorem 3 we conclude that for each $y \in E$ all circuits containing x have the same cardinality. This is so since if C, D were circuits containing y with $|D| < |C|$, we could choose a basis B with $C\backslash\{y\} \subseteq B$. For this basis B, $C_y = C$.

Since $\mathbf{M}$ is nonseparable each pair of elements of E lies in a circuit. Thus all circuits of $\mathbf{M}$ have the same cardinality, say k. Let B be any basis of $\mathbf{M}$. Let $x, y \in E \backslash B$ and choose a circuit D containing x and y. If $C_x \backslash \{x\} \neq C_y \backslash \{y\}$, then

$$\left| \bigcup_{z \in D} C_z \cap B \right| \geq |(C_x \backslash \{x\}) \cup (C_y \backslash \{y\})| \geq |C_x| = |D|$$

and as in the proof of Theorem 3 we conclude that $\mathbf{M} \neq \mathbf{T}(C_e^*: e \in B)$. Thus $C_x \backslash \{x\} = C_y \backslash \{y\}$ for all $x, y \in E \backslash B$. But since $\mathbf{M}$ is nonseparable, this readily implies that $C_x \backslash \{x\} = B$ for all $x \in E \backslash B$. In particular $k = r + 1$.

Now let F be any r element subset of E. Since all circuits of $\mathbf{M}$ have cardinality $r + 1$, A is a basis of $\mathbf{M}$. Hence $\mathbf{M} = \mathcal{P}_r(E)$.

Corollary 6. *If $\mathbf{M}$ is a matroid of rank r on E such that $\mathbf{M} = \mathbf{M}(C_e^*: e \in B)$ for all bases B, then there is a partition $E_1, \cdots, E_t$ of E with $\mathbf{M} = \mathcal{P}_{r_1}(E_1) \oplus \cdots \oplus \mathcal{P}_{r_t}(E_t)$ where $r = r_1 + \cdots + r_t$.*

Example. Let $\mathbf{M}$ be the cycle matroid of the graph in Figure 2.

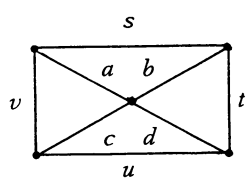

Figure 2

With respect to the basis $B = \{a, b, c, d\}$, for each $x \notin B$, C_x has smaller cardinality (namely 3) than any other circuit containing x. Thus the possibility that $\mathbf{M} = \mathbf{T}(C_x^*: x \in B)$ is not ruled out by Theorem 3. But if $D = \{s, t, u, v\}$, then D is a circuit of $\mathbf{M}$ and $D \in \mathbf{T}(C_x^*: x \in B)$. (This is most readily checked by using Theorem 1.) Thus $\mathbf{M} \neq \mathbf{T}(C_x^*: x \in B)$.

REFERENCES

1. J. A. Bondy and D. J. A. Welsh, *Some results on transversal matroids and constructions for identically self-dual matroids*, Quart J. Math. Oxford Ser. (2) 22 (1971), 435–451. MR 44 #3899.

2. H. Crapo and G. C. Rota, *On the foundations of combinatorial theory: Combinatorial geometries*, Preliminary edition, M.I.T. Press, Cambridge, Mass., 1970. MR 45 #74.

3. J. Edmonds and D. R. Fulkerson, *Transversals and matroid partition*, J. Res. Nat. Bur. Standards Sect. B 69 B (1965), 147–153. MR 32 #5531.

4. F. Harary and D. Welsh, *Matroids versus graphs*, The Many Facets of Graph Theory (Proc. Conf., Western Mich. Univ., Kalamazoo, Mich., 1968), Springer, Berlin, 1969, pp. 155–170. MR 41 #8267.

5. M. Las Vergnas, *Sur les systèmes de représentants distincts d'une famille d'ensembles*, C. R. Acad. Sci. Paris Sér. A-B 270 (1970), A501-A503. MR 42 #1663.

6. L. Mirsky, *Transversal theory. An account of some aspects of combinatorial mathematics*, Math. in Science and Engineering, vol. 75, Academic Press, New York, 1971. MR 44 #87.

7. H. Whitney, *On the abstract properties of linear dependence*, Amer. J. Math. 57 (1935), 509–553.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706